Graded rings over K3s

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in memory of Hideyuki Matsumura

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This lecture is based on joint work with Selma Altınok. The object of study is a polarised K3 (X, D). Here X is a K3 having at worst Du Val singularities, and D an ample Weil divisor. One passes back and forwards between (X, D) and the graded ring

$$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)), \quad \text{with} \quad X = \operatorname{Proj} R(X, D).$$

These ideas have many applications to singularity theory, $\mathbb{Q}$-Fano 3-folds, mirror symmetry, and speculation on the structure of Gorenstein rings in small codimension.

Professor Hideyuki Matsumura was a family friend since his visit to Warwick in 1983 (during which, among other things, he helped me with several points of English grammar in the translation of his textbook). I had the greatest respect for him as a mathematician and teacher, a Christian, a keen mountaineer, and an unusually well informed liberal political thinker. This lecture is dedicated in warm gratitude to his memory.

Contents

§1. Numerical type 2

§2. Hilbert function 3

§3. Moduli, lattice 3

§4. How to make lists 5

§5. Pfaffians 7

§6. Gorenstein rings in codimension 4 and unprojecting 9

1 Numerical type

The *numerical data* of (X, D) is an expression of the form

$$D^2 = \Gamma^2 + \sum_i \frac{a_i(r_i - a_i)}{r_i},$$

where $\Gamma^2 \in 2\mathbb{Z}$, and the fractional contributions correspond to a *basket* of cyclic quotient singularities. (X, D) is *quasismooth* if X has at worst cyclic quotient singularities and D generates each local class group. Then each singularity is a cyclic quotient singularity of type $\frac{1}{r}(1, -1)$, and $\mathcal{O}_X(D)$ is locally of some type c coprime to r (that is, isomomorphic to the eigensheaf of ε^c); we say that $\frac{1}{r}(c, -c)$ is the type of the (polarised) singularity, but the numerical data $a(r-a)/r$ records r and the inverse of c modulo r, that is, the a with $1 \leq a \leq r - 1$ and $ac \equiv 1$. Quasismooth is equivalent to the condition that the affine cone $\operatorname{Spec} R(X, D)$ is nonsingular. (The quasismooth case is generic, and we can restrict to it for most purposes. We will see in §6 below that the case when X is not quasismooth is also interesting.)

Example $f = f_{44}(x, y, z, t)$ is the general weighted homogeneous polynomial of degree 44 in variables of weights $4, 5, 13, 22$; then $X = \operatorname{Proj} k[x, y, z, t]/f$ is the weighted hypersurface $X_{44} \subset \mathbb{P}(4, 5, 13, 22)$, and has numerical data

$$D^2 = \frac{44}{4 \cdot 5 \cdot 13 \cdot 22} = \frac{1}{130} = -4 + \frac{1}{2} + \frac{2 \cdot 3}{5} + \frac{3 \cdot 10}{13};$$

it is quasismooth, with

$$\Gamma^2 = -4, \text{ and singularities } \begin{cases} 1 \times \frac{1}{2}(1,1) & \text{on the } xt \text{ line } \{(x,0,0,t)\}; \\ \frac{1}{5}(2,3) & \text{at the } y \text{ point } (0,1,0,0); \\ \frac{1}{13}(4,9) & \text{at the } z \text{ point } (0,0,1,0). \end{cases}$$

2 Hilbert function

Write $P_n = h^0(X, nD) = \dim H^0(X, \mathcal{O}_X(nD))$. Then P_n is given by the RR formula of [YPG], §8: $P_0 = 1$ and

$$P_n = 2 + \frac{1}{2}n^2D^2 - \sum_{\text{basket}} \frac{\overline{na}(r - \overline{na})}{2r} \quad \text{for } n \geq 1, \tag{1}$$

where $\overline{x}$ = smallest residue mod r and the sum takes place over the basket $r = r_i, a = a_i$ for $i = 1, 2, \dots$. These terms are quite tricky to calculate individually, but they can be handled together as the rational function

$$P(t) = \sum_{n \geq 0} P_n t^n = \frac{1+t}{1-t} + \frac{t+t^2}{(1-t)^3}\frac{D^2}{2} - \sum \frac{1}{1-t^r} \sum_{i=0}^{r} \frac{\overline{ia}(r - \overline{ia})}{2r} t^i. \tag{2}$$

In this expression, the first term is $1 + 2t + 2t^2 + \cdots$, the second is $D^2/2$ times $t + 4t^2 + 9t^3 + \cdots$, and for each element of the basket, the factor $\frac{1}{(1-t^r)}$ just repeats periodically the function $(\overline{ia}(r - \overline{ia}))/2r$ (which is a symmetric crenallation over the interval $[0, r]$, zero at the endpoints).

Example $X_{44} \subset \mathbb{P}(4, 5, 13, 22)$ has

$$\begin{aligned}
&\frac{1 - t^{44}}{(1-t^4)(1-t^5)(1-t^{13})(1-t^{22})} \\
&\quad = \frac{1+t}{1-t} + \frac{1}{260}\frac{t+t^2}{(1-t)^3} - \frac{1}{1-t^2}\frac{t}{4} - \frac{1}{1-t^5}\frac{6t + 4t^2 + 4t^3 + 6t^4}{10} \\
&\qquad - \frac{1}{1-t^{13}}\frac{30t + 42t^2 + 36t^3 + 12t^4 + 22t^5 + 40t^6 + (\text{sym})}{26}.
\end{aligned}$$

For example, the coefficient of t on the right-hand is 0, and the computation is almost the same as in the above formula for $D^2 = \frac{1}{130}$.

3 Moduli, lattice

K3s and their moduli appear in many areas of math. Very recent applications in mirror symmetry involve the moduli of K3s with a fixed sublattice in $\operatorname{Pic} S$. For this lecture, I give the statements of this section as conjectures.

Conjecture 3.1 *X, D has a quasismooth small deformation X_t, D_t.*

Proof The local statement is proved in [YPG], §8: a Du Val singularity X with a Weil divisor D deforms to a number of cyclic quotient singularities polarised by D. Also, the global deformations of X are unobstructed and map surjectively to local deformations of the singularities $P \in X, \mathcal{O}_{X,P}(D)$ (to be rigorous, this argument needs a little more care). Q.E.D.

The numerical data of X, D can be seen on the minimal resolution of singularities $S \to X$ as a *almost star shaped* graph. That is, a central node Γ, together with a number of chains of length $r-1$ of -2-curves, with Γ joined to the ath node of the chain. Each chain is weighted with two arithmetic progressions up to the ath node:

$$\frac{1}{r}\left(\begin{array}{ccccccccccccc} a & & 2a & & \dots & & (r-a)a & & \dots & & r-a \\ \bigcirc & — & \bigcirc & — & \dots & — & \bigcirc & — & \dots & — & \bigcirc \\ & & & & & & | & & & & \\ & & & & & & \Gamma & & & & \end{array}\right)$$

It is easy to see that this is the only way of weighting the graph so that $DE_i = 0$ for all the exceptional curves. (The picture shows only one branch out of Γ.)

Lemma 3.2 *The numerical data of a polarised K3 X, D satisfies*

(a) Γ^2 *is even and* ≥ -4*;*

(b) $\sum(r-1) \leq 19$ *(in char* 0*)*

Proof The divisors Γ and E_i are linearly independent in $\operatorname{Pic} S$, so that (b) comes from $\rho(S) \leq 20$. To prove (a), standard messing around with vanishing and the resolution of rational singularities gives

$$H^i(S, \mathcal{O}_S(\Gamma)) = H^i(X, \mathcal{O}_X(D)) = 0 \quad \text{for } i = 1, 2,$$

so that RR on S gives $H^0(S, \mathcal{O}_S(\Gamma)) = 2 + \frac{1}{2}\Gamma^2$.

Conjecture 3.3 *(I) Every numerical type satisfying (a) and (b) corresponds to a family X, D of polarised K3s, forming an irreducible moduli space of dimension* $19 - \sum(r-1)$.

(II) Γ *and* E_i *are linearly independent and generate a sublattice* L_0 *of rank* $1+\sum(r-1)$ *of* $\operatorname{Pic} S$*; moreover, the primitive sublattice* L *corresponding to* L_0 *can be easily determined in terms of submultiples of* D*.*

(III) If the ring $R(X, D)$ *has a simple description in commutative algebra (say, as a hypersurface or a codimension 2 complete intersection), then* L_0 *is primitive in* $\operatorname{Pic} S$*.*

For the Famous 95 K3 hypersurfaces, (III) has been proved by Belcastro and Dolgachev [Belcastro], by a case-by-case computation. By analogy with the familiar formula for the discriminant of the $T_{p,q,r}$ lattices, Dolgachev points out the formula

$$\operatorname{discrimant}(L_0) = \prod r \times \left(\Gamma^2 - \sum \frac{a(r-a)}{r}\right).$$

One could attempt to prove (III) directly as follows. Consider the 3-fold cone $C_X \subset \mathbb{C}^4$ defined by $f_d(x, y, z, t) = 0$. This is nonsingular outside the origin, so by Lefschetz theory, the inclusion $C_X \setminus 0 \hookrightarrow \mathbb{C}^4 \setminus 0$ is 2-connected. In particular, $C_X \setminus 0$ is simply connected. But a nontrivial inclusion of finite index $L_0 \subset L \subset \operatorname{Pic} S$ would give a branched cover of X (branched only over the cyclic quotient singularities), therefore an unbranched cover of $C_X \setminus 0$.

4 How to make lists

The lists we have at present are the following

1. Reid, 1979: The Famous 95 hypersurfaces [Fletcher], pp. 31–32, rediscovered by Yonemura 1989 and subsequently by many others.

2. Fletcher, 1988: The 84 codimension 2 complete intersections [Fletcher], pp. 34–35. (He also finds a unique $\mathbb{Q}$-Fano 3-fold

$$X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7) \quad \text{having } |-K_X| = \emptyset,$$

in codimension 2, which has no K3 hyperplane sections.)

3. It is easy to see that there is only one weighted codimension 3 complete intersection, namely the intersection of 3 quadrics $X_{2,2,2} \subset \mathbb{P}^5$.

4. Altınok, 1996: There are 69 codimension 3 $X \subset \mathbb{P}(a_1, \ldots, a_6)$ defined by Pfaffians of 5×5 skewsymmetric matrix. (This list is probably 99% accurate.)

5. Altınok, 1996: A raw list of 93 candidates for codimension 4 rings. (We guess that this list is probably about 80% accurate.)

There are at least 3 quite different methods to obtain the list of Famous 95. Here I describe Altınok's *Hilbert function method*, which extends and improves the "Table method" of Fletcher and Reid. We work with the Hilbert series $P(t)$ (2); if the ring $R(X, D)$ has a nonzero element x_1 of degree a_1, then $(1-t^{a_1})P(t)$ is the Hilbert series of the graded ring built over the section $(x_1 = 0)$ of X; similarly, if $x_1, x_2 \in R(X, D)$ form a regular sequence then $(1 - t^{a_1})(1 - t^{a_2})P(t)$ is the Hilbert series corresponding to the codimension 2 complete intersection $x_1 = x_2 = 0$ in X. I hope you can recognise $(1 - t^{a_1})(1 - t^{a_2})$ as a Koszul complex.

Altınok develops a practicable method of finding degrees a_1, a_2, a_3 for element x_1, x_2, x_3 such that

$$(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})P(t) = \sum_{i=0}^{d} c_i t^i \tag{3}$$

is a finite polynomial, with $c_i \geq 0$, and such that the a_i are not too big. This is the Hilbert function equivalent of passing to the Artinian quotient ring of $R(X, D)$, cutting by a regular sequence x_1, x_2, x_3. This means

(i) For (3) to be a polynomial, the 3 factors $(1 - t^{a_i})$ must kill the denominators $1 - t^r$ of (2), that is, every r divides one of the a_i. (We call this "killing the periodicity".)

(ii) For (3) to be positive means in particular that we only introduce x_1 in degree a_1 if $P(a_1) > 0$, then x_2 in degree a_2 if $P(a_2) - P(a_2 - a_1) > 0$, etc.

(iii) "Not too big" means that we can hope that (3) is the Hilbert function of an Artinian ring of small codimension. For example, we can certainly get an Artinian quotient by cutting by x_1, x_2, x_3 of some degree a large common multiple of the indexes r, but this is too big to be useful.

This calculation needs trial-and-error, and is hard to automate, but the smallest values a_1, a_2, a_3 compatible with (i) and (ii) usually work.

To make a list of possible rings $R(X,D)$ of given codimension, we do the following: there are a few thousand possible baskets of singularities $\{r,a\}$ on K3s, which we write out in some order. For each basket, Γ^2 is an even number ≥ -4 chosen so that $0 \leq P_1 \leq \text{codim} + 3$. Now the condition that the ring $R(X,D)$ has at most codim $+3$ generators imposes very many necessary conditions on the numerical data of X: for example, Altınok proves that a singularity of type $\frac{1}{r}(a,-a)$ in the basket implies that the ring has 3 generators of degrees $\equiv 0, a, r-a$ modulo r. This rapidly cuts down the list to a fairly small number of plausible candidates. Our experience is that a plausible candidate usually gets through (contrary to the current sad state of the job market).

5 Pfaffians

A typical example (No. 18 in the current draft of Altınok's codimension 3 list):

$$D^2 = -2 + 2 \times \frac{1}{2} + \frac{3 \cdot 4}{7} = \frac{5}{7}\,.$$

It is easy to see that $P_1 = 1$ and $P_2 = 3$, and we must have at least a generator of degree 2 and one of degree 7 to kill the periodicity. Note that 2 is the inverse of 4 mod 7, so that the type of the singularity is $\frac{1}{7}(2,5)$. The simplest solution is to take x, y_1, u of degree $1, 2, 7$. This gives

$$\begin{aligned} P(t) = {} & \frac{1+t}{1-t} + \frac{t+t^2}{(1-t)^3}\frac{5}{14} \\ & - 2 \times \frac{1}{1-t^2}\frac{t}{4} - \frac{1}{1-t^7}\frac{12t + 6t^2 + 10t^3 + 10t^4 + 6t^5 + 12t^6}{14}\,, \end{aligned}$$

and after a little simplifying

$$(1-t)(1-t^2)(1-t^7)P(t) = 1 + t^2 + t^3 + t^4 + 2t^5 + t^6 + t^7 + t^8 + t^{10}.$$

(Access to a small computer running Maple is an advantage if you intend to perform these calculations on an industrial scale.) This looks like the Hilbert series of an Artinian ring with further generators y_2, z, t in degree $2, 3, 5$, so that the plausible candidate is $X \subset \mathbb{P}(1,2,2,3,5,7)$. To find the structure of its resolution, multiply again by $(1-t^2)(1-t^3)(1-t^5)$, to give

$$\begin{aligned}(1-t)(1-t^2)^2(1-t^3)(1-t^5)(1-t^7)P(t) &= \\ 1-t^6-t^7-t^8-t^9-t^{10}+t^{10}+t^{11}+t^{12}+t^{13}+t^{14}-t^{20}.\end{aligned}$$

The numerical shape of the complex resolving $R(X,D)$ over the polynomial ring $\mathcal{O} = k[x,y_1,y_2,z,t,u]$ is therefore

$$\mathcal{O} \leftarrow \mathcal{O}(-6,-7,-8,-9,-10) \leftarrow \mathcal{O}(-10,-11,-12,-13,-14) \leftarrow \mathcal{O}(-20) \leftarrow 0$$

In other words, we expect 5 relations in degrees $6,7,8,9,10$ and 5 syzygies in degrees $10,11,12,13,14$. Note the Gorenstein symmetry $n \mapsto k-n$ where

$$k = 20 = 1+2+2+3+5+7$$

corresponds to the canonical class of $\mathbb{P}(1,2,2,3,5,7)$. The shape of the polynomial, together with the Buchsbaum–Eisenbud theorem on Gorenstein rings in codimension 3 [B–E] instructs us to look for the equations as the Pfaffians of a skew symmetric matrix M with degrees

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ & 2 & 3 & 4 & 5 \\ & & 4 & 5 & 6 \\ & -\mathrm{sym} & & 6 & 7 \\ & & & & 8 \end{pmatrix}.$$

We can check that for general entries in M, the Pfaffians[1] of M define a nonsingular K3 surface X. For example, consider the u point $(0,0,\dots,1)$. Since only one entry m_{45} of M has degree ≥ 7, the variable u can only appear in this term, and I can assume that $m_{45} = u$. It is involved in exactly 3 Pfaffians, namely $P(23.45), P(13.45), P(12.45)$. Thus if $m_{23} = z$, $m_{13} = y_1$, $m_{12} = x$, these 3 equations give $ux = \cdots$, $uy_1 = \cdots$ and $uz = \cdots$. Thus by

[1]**Tutorial.** I hope you remember what a Pfaffian is:

$$P(ij.kl) = m_{ij}m_{kl} - m_{ik}m_{jl} + m_{il}m_{jk}.$$

The formula is not much harder to remember than that for a 2×2 determinant. If in doubt, check as an easy exercise that the diagonal maximal minor is $P(ij.kl)^2$, and that every 4×4 minor of M is a product of two Pfaffians.

the implicit function theorem, near the u point (where we will set $u = 1$), the 3 variables x, y_1, z are functions of other variables, leaving $\mathbb{C}^2_{y_2,t}$ divided by the action $\frac{1}{7}(2,5)$. In more detail, the affine cone defined by the 5 Pfaffians is nonsingular along the u axis, but when I pass to Proj by taking the $\mathbb{C}^*$ quotient, the u axis is fixed by the subgroup $\mu_7 \subset \mathbb{C}^*$, which acts by $\frac{1}{7}(2,5)$ on the transverse variables y_2, t.

I omit the remainder of the nonsingularity calculation, which is similar, although not necessarily without its tricky aspects.

Exercises Here is a short sample of Altınok's codimension 3 list (No. 7, 46, 47, 49, 50) to try out as exercises:

$$\begin{aligned}
D^2 = 0 + \frac{1\cdot 5}{6} = \frac{5}{6} &\implies X(6,7,8,9,10) \subset \mathbb{P}(1,1,3,4,5,6)\\
D^2 = -2 + 3\times\frac{1}{2} + \frac{1\cdot 2}{3} + \frac{1\cdot 3}{4} = \frac{11}{12} &\implies X(5,6,6,6,7) \subset \mathbb{P}(1,2,2,3,3,4)\\
D^2 = -2 + 2\times\frac{1}{2} + \frac{1\cdot 2}{3} + \frac{2\cdot 3}{5} = \frac{13}{15} &\implies X(5,6,6,7,8) \subset \mathbb{P}(1,2,2,3,3,5)\\
D^2 = -4 + \frac{1}{2} + \frac{1\cdot 5}{6} + \frac{5\cdot 6}{11} = \frac{2}{33} &\implies X(14,15,16,17,18) \subset \mathbb{P}(2,5,6,7,9,11)\\
D^2 = -4 + 2\times\frac{1}{2} + \frac{2\cdot 5}{7} + \frac{3\cdot 4}{7} = \frac{1}{7} &\implies X(9,10,11,12,14) \subset \mathbb{P}(2,3,4,5,7,7)
\end{aligned}$$

6 Gorenstein rings in codimension 4 and unprojecting

The problem in this section is harder because there is no structure theorem for Gorenstein rings in codimension 4 comparable to the Buchsbaum–Eisenbud theorem for codimension 3. However, there is a considerable body of experience suggesting that Gorenstein rings in codimension 4 belong to one of a small number of tightly controlled structures, and Altınok's list contains dozens more cases. In the remainder of the talk, I treat only a couple of cases out of the many which supports the following slogan:

Gorenstein rings in small codimension are about unprojecting.

Example 1 Consider $D^2 = 8 + \frac{1}{2}(1\cdot 1)$ (omitted in the current draft of Altınok's codimension 4 list). This means that X is a K3 with just one ordinary double point. The linear system $|D|$ (of Weil divisors) blows up the node to a -2-curve on the nonsingular model S, and embeds S as a complete intersection $S_{2,2,2} \subset \mathbb{P}^5$ containing a line E.

Choose coordinates so that $E : (x_1 = x_2 = x_3 = x_4 = 0)$. The equations of the 3 quadrics can be written

$$Q_i : \sum_{j=1}^{4} n_{ij} x_j = 0 \quad \text{for } i = 1, 2, 3, \tag{4}$$

where $N = n_{ij}$ is a 3×4 matrix. The $\mathbb{Q}$-divisor on S is $\Gamma + \frac{1}{2}E$. In degree 2, there is a new generator $y \in H^0(S, 2\Gamma + E)$ with pole along E, which I interpret as a homomorphism $y\colon \mathcal{I}_E \to \mathcal{O}_S(2\Gamma)$, so that

$$yx_i \in H^0(S, \mathcal{O}_S(3\Gamma)) \quad \text{for } i = 1, \dots, 4.$$

Cramér's rule from linear algebra says that we can solve (4) to get $yx_i = i$th 3×3 minor of N. Using this, you see that $R(X, D) = k[x_1, \dots, x_6, y]/I_X$ is the Gorenstein ring in codimension 4 defined by the 7 equations

$$N\mathbf{x} = 0, \quad y\mathbf{x} = \bigwedge^3 N,$$

where N is a 4×3 matrix and $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Compare [K–M].

Here $S \subset \mathbb{P}^5$ is obtained by projecting $X \subset \mathbb{P}(1^6, 2)$ from the y point $(0, \dots, 0, 1)$; the inverse construction is to *unproject* (contract) $E \subset S$ to give X.

Unprojection I observe some features of Example 1, which I take as a working definition of *unprojection*. The construction starts from $E \subset S \subset \mathbb{P}$, where $S \subset \mathbb{P}$ is a projectively Gorenstein subscheme in codimension 3, containing a subscheme such that $E \subset \mathbb{P}$ is a projectively Gorenstein subscheme in codimension 4 (in Example 1, both S and E are complete intersections). The adjunction formula gives

$$\omega_E = \mathcal{E}xt^1(\mathcal{O}_E, \omega_S) = \mathcal{H}om(\mathcal{I}_E, \omega_S)/\omega_S.$$

Together with the projectively Gorenstein assumptions

$$\omega_S = \mathcal{O}_S(k_S) \quad \text{and} \quad \omega_E = \mathcal{O}_E(k_E) \quad \text{with } k_S, k_E \in \mathbb{Z},$$

this means that $\mathcal{H}om(\mathcal{I}_E, \mathcal{O}_S)/\mathcal{O}_S = \mathcal{O}_E(l)$ is just a Serre twist of $\mathcal{O}_E$ (with $l = k_S - k_E$), so that there exists a homomorphism $y\colon \mathcal{I}_E \to \mathcal{O}_S(l)$ which bases $\mathcal{H}om(\mathcal{I}_E, \omega_S)$ at every point of E.

Now the following construction seems to make sense in this generality, provided that $l = k_S - k_E > 0$: write $A = k[\mathbb{P}] = k[x_1, \dots, x_N]$ for the homogeneous coordinate ring of $\mathbb{P}$, and adjoin a new generator y of degree $l = k_S - k_E$ to give $B = A[y]$ and a new weighted projective space $\mathbb{P}' = \operatorname{Proj} B$. Moreover, write $I_S \subset A$ for the homogeneous ideal defining $S \subset \mathbb{P}$, and define a new ideal $I_X \subset B$ as follows: for each generator $F_i \in I_E$, represent $y(F_i) \in k[S]$ by a polynomial $G_i \in A$, and set

$$I_X = \Big(I_S, \{yF_i - G_i\}\Big).$$

Then define $X \subset \mathbb{P}'$ to be the *unprojection* of E in S. I believe that it is a projectively Gorenstein subscheme (possibly under mild additional assumptions).

The Buchsbaum–Eisenbud theorem as unprojecting Let $M = \{m_{ij}\}$ be the generic $(2k+1) \times (2k+1)$ skewsymmetric matrix, and $X \subset \mathbb{P}^N$ (where $N = \binom{2k+1}{2} - 1$) the projectively Gorenstein subvariety of codimension 3 defined by the ideal of $(2k) \times (2k)$ Pfaffians of M.

Obviously exactly 2 of the Pfaffians defining X do not involve m_{12}, namely P_1, P_2 obtained by deleting row and column 1 or 2; thus eliminating m_{12} projects X to a codimension 2 complete intersection $S : (P_1 = P_2 = 0) \subset \mathbb{P}^{N-1}$. Moreover, expanding P_1, P_2 along their top row expresses them as linear combinations of the $(2k-2) \times (2k-2)$ Pfaffians of the bottom $(2k-1) \times (2k-1)$ block of M. Thus S contains a projectively Gorenstein subvariety of codimension 3 E, and it is easy to see that X can be recovered by unprojecting $E \subset S$, as described quite recently.

Example 2 The numerical data

$$D^2 = -2 + 3 \times \frac{1}{2}(1 \cdot 1) + \frac{1}{7}(2 \cdot 5)$$

(No. 12 in the current draft of Altınok's codimension 4 list) leads to a beautiful case study in unprojecting. The usual Hilbert function calculation gives the plausible candidate $X \subset \mathbb{P}(1, 2, 2, 3, 3, 4, 7)$ with Hilbert series

$$1 - t^5 - 3t^6 - t^7 + t^9 + 2t^{10} + 2t^{11} + 2t^{12} + t^{13} - t^{15} - 3t^{16} - t^{17} + t^{22}.$$

You probably can't guess at once from this that in fact you need

$$\begin{aligned}&9 \text{ relations of degrees } 5, 6^3, 7, 8, 9^2, 10 \text{ and}\\ &16 \text{ syzygies of degrees } 8, 9^3, 10^3, 11^2, 12^3, 13^3, 14.\end{aligned}$$

(The numerical shape of the resolution is then determined by Gorenstein symmetry about $k = 22$.)

To see this, work with the two subrings of $R(X, D) = k[x, y_1, y_2, z_1, z_2, t, u]/I_X$ generated by the elements of degree ≤ 3 and degree ≤ 4. Let $\varphi\colon S \to X$ be the minimal resolution, $3 \times A_i$ the -2-curves over the nodes $\frac{1}{2}(1, 1)$, and $E_1, \dots, E_6$ the chain of curves over the index 7 point. I write a combination $a_1E_1 + a_2E_2 + \cdots$ as a vector $(a_1, a_2, \dots)$. Thus

$$\varphi^* D = \Gamma + \frac{1}{2}\sum A_i + \frac{1}{7}(5, 10, 8, 6, 4, 2).$$

It turns out that the subring of $R(X, D)$ generated by x, y_1, y_2, z_1, z_2 defines a morphism of S to a K3 codimension 2 complete intersection

$$\varphi_{2,3}\colon S \to Y_{5,6} \subset \mathbb{P}(1, 2, 2, 3, 3)$$

([Fletcher], List II.3.8, p. 34, No. 12). $\varphi_{2,3}$ corresponds to the $\mathbb{Q}$-divisor $\Gamma + \frac{1}{2}\sum A_i + \frac{1}{6}(4, 8, 6, 4, 2, 0)$, which contracts all the lines $E_1, \dots, E_5$ except E_6 to a singularity of type $\frac{1}{6}(1, -1)$ at the z_1 point of $\mathbb{P}(1, 2, 2, 3, 3)$, and maps E_6 to the xz_1 line $\overline{E}_6 \subset Y_{5,6}$. Note that $Y_{5,6}$ is not general, and not quasismooth at the z_1 point (because D is not a local generator of the class group).

Next, the subring of $R(X, D)$ generated by x, y_1, y_2, z_1, z_2, t defines a morphism of S to a Pfaffian K3

$$\varphi_{3,4}\colon S \to Z(5, 6, 6, 6, 7) \subset \mathbb{P}(1, 2, 2, 3, 3, 4)$$

(No. 46 in Altınok's codimension 3 list), whose equations are the Pfaffians of a matrix of degrees

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 3 \\ & 3 & 3 & 3 & 4 \\ & & 3 & 3 & 4 \\ & -\text{sym} & & 3 & 4 \\ & & & & 5 \end{pmatrix}.$$

$\varphi_{3,4}$ corresponds to the $\mathbb{Q}$-divisor $\Gamma + \frac{1}{2}\sum A_i + \frac{1}{12}(8,16,12,9,6,3)$, which contracts all the exceptional curves of S except E_3, and maps E_3 to the z_2t axis $\overline{E}_3 \subset Z \subset \mathbb{P}(1,2,2,3,3,4)$.

The beautiful thing is that each step from Y to Z and from Z to X is an unprojection. First, the Pfaffian Z is obtained by unprojecting the xz_1 line $\overline{E}_6 \subset Y$: it is the codimension 3 complete intersection $y_1 = y_2 = z_2 = 0$ contained in a codimension 2 complete intersection. The element $t \in \mathcal{H}om(\mathcal{I}_{\overline{E}_6}, \omega_Y(4))$ which restricts to a basis of $\omega_{\overline{E}_6}(4)$ is the new generator of degree 4, and satisfies 3 new equations

$$ty_1 = \cdots, \quad ty_2 = \cdots, \quad tz_2 = \cdots$$

Next, the codimension 4 variety $X \subset \mathbb{P}(1,2,2,3,3,4,7)$ is obtained by unprojecting the line the z_2t line $\overline{E}_3 \subset Z$: it is the codimension 4 complete intersection $x = y_1 = y_2 = z_1 = 0$ contained in a 5×5 Pfaffian. The element $t \in \mathcal{H}om(\mathcal{I}_{\overline{E}_3}, \omega_Z(7))$ which restricts to a basis of $\omega_{\overline{E}_3}(7)$ is the new generator of degree 7, and satisfies 4 new equations

$$ux = \cdots, \quad uy_1 = \cdots, \quad uy_2 = \cdots, \quad uz_1 = \cdots .$$

Remarks (1) The question of how a known codimension 4 Gorenstein subvariety is to be contained in a codimension 3 Pfaffian is of course obstructed. Thus unprojecting does not provide an automatic answer to the question of structure theory of codimension 4 Gorenstein rings.

(2) At times I have heard David Eisenbud conjecture that codimension 4 Gorenstein rings are to do with liaisons – there is no contradiction with my ideas here, since unprojecting can of course be interpreted as a special case of liaisons.

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