Surfaces with $p_g = 3$, $K^2 = 4$ according to E. Horikawa and D. Dicks

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Abstract

This is the text of a lecture given at 2 workshops at the Univ. of Utah in Nov 1989 and the Univ. of Tokyo in Dec 1989, an introduction to the Warwick thesis of Duncan Dicks [D1], [D2]. The aim is to study a class of surfaces of general type (in practice necessarily regular, that is, $q = 0$) in terms of the canonical ring. This leads to lots of algebra, deformation theory, and very interesting questions on how to recover the geometry from the algebra. I should point out that the choice of the class of surfaces to study is rather delicate: the two classes that have been studied in great detail are the numerical quintics $p_g = 4$, $K^2 = 5$ [H1], [R2] and $p_g = 3$, $K^2 = 4$. In both these cases detailed results were obtained by Horikawa using geometric and analytic arguments [H1], [H2]. But if we change the invariants, e.g., to $p_g = 4$, $K^2 = 7$, then the calculations become very much bigger, and it is unlikely that a similar complete analysis is possible with present technology.

Two possible generalisations of this material are discussed at the end of Section 5 (in case anyone want a PhD problem in this area). There are, unfortunately, errors of detail in the computations in all the papers [G], [R2], [D1], [D2], and implementing the computer algebra algorithm of [R2], Section 6 to give reliable results in reasonable generality remains a challenge.

Contents

Set-up

Let X be a canonical surface of general type with $q = 0$, and $C \in |K|$ a general canonical curve (allowed to be singular). Write

$$R(S) = R(X, K_X) = \bigoplus H^0(X, nK_X)$$

and $R(C) = R(C, K_{X|C}) = \bigoplus H^0(C, nK_{X|C})$ for the graded rings. The technique is to write out generators and relations for $R(C)$, then $R(X)$ if possible; generalities on this procedure, and some examples are given in [R2].

1 Geometry: The Horikawa analysis

Let S be a surface of general type with $p_g = 3$, $q = 0$ and $K^2 = 4$; as usual S is nonsingular with K_S nef and big, but if I prefer I can work with the canonical model X of S, which has possibly Du Val singularities but K_X ample. The following analysis is the elementary part of Horikawa's work, and works nicely because K^2 is small compared to p_g: write

$$|K_S| = \begin{Bmatrix} \text{codim}\,1 \\ \text{base locus} \end{Bmatrix} + \text{movable part } M$$

and

$$\text{blowup of } |M| = \begin{Bmatrix} \text{codim} \ge 2 \\ \text{base locus} \end{Bmatrix} + \text{free } F.$$

Let $\varphi_{K_S}\colon S \dashrightarrow \mathbb{P}^2$ be the rational map defined by $|K_S|$. Then

$$4 = K_S^2 \ge \cdots \ge F^2 = \deg\varphi \cdot \deg\varphi(S).$$

It turns out that $\varphi(S) = \mathbb{P}^2$, so that $F^2 \ge 2$; every component of the base locus makes a positive difference to the "...", so I get the next result:

Theorem 1.1 *One of the following holds:*

(I) $|K_S|$ *is free and* $\deg\varphi = 4$.

(II) $|K_S|$ *has* 1 *transverse base point* P *and* $\deg\varphi = 3$.

(III) $|K_S|$ *has* 2 *transverse base points* $P_1 \ne P_2$ *and* $\deg\varphi = 2$.

(III_a, III_b) *On the minimal model S, $|K_S|$ has a base -2-cycle Z; on X, $|K_X|$ has a base Du Val singularity.*

In the two last cases, $|K_S| = |\Gamma| + Z$, where $|\Gamma|$ is a free linear system with $\Gamma^2 = 2$, $Z^2 = -2$, and $\Gamma_{|\Gamma} = g^1_2$. The general divisor of $|K_S|$ is of the form $\Gamma + Z$, where Γ meets Z in 2 distinct points in case (III_a) and 2 infinitely near points in (III_b).

Proposition 1.2 *Case (III_b) does not occur.*

Proof I claim that if $\Gamma \cap Z = 2P$ then

$$\mathcal{O}_{\Gamma+Z}(\Gamma + Z) \cong \mathcal{O}_{\Gamma+Z}(2g^1_2);$$

this contradicts $H^0(\mathcal{O}_S(K_S)) \twoheadrightarrow H^0(\Gamma + Z, \mathcal{O}_{\Gamma+Z}(K_S))$. To prove the claim, note that

$$\mathcal{O}_{\Gamma+Z}(2\Gamma + 2Z) \cong \mathcal{O}_{\Gamma+Z}(K_{\Gamma+Z}) \cong \mathcal{O}_{\Gamma+Z}(4g^1_2).$$

Therefore $\mathcal{O}_{\Gamma+Z}(\Gamma + Z - 2g^1_2)$ is a 2-torsion class in $\mathrm{Pic}(\Gamma + Z)$. But since $\Gamma \cap Z = 2P$ it follows that $\ker\{\mathrm{Pic}^0(\Gamma + Z) \to \mathrm{Pic}^0 \Gamma\} \cong \mathbb{G}_a = k^+$, and this group has no 2-torsion. Q.E.D.

2 Algebra, easy cases

In this section I treat cases (I) and (II). These are in many ways ideal examples, since the algebra is straightforward, and has direct geometric applications to the study of individual surfaces and to their deformations.

Theorem 2.1 *In case (I),*

$$R(C) = k[x_1, x_2, y_1, y_2]/(f_4, g_4)$$

and

$$R(X) = k[x_0, x_1, x_2, y_1, y_2]/(F_4, G_4),$$

so that $X = X_{4,4} \subset \mathbb{P}(1^3, 2^2)$ is a complete intersection in a weighted projective space (here and below, variables x_i, y_i, z_i have weights 1, 2, 3 respectively).

In case (II), $R(C) = k[x_1, x_2, y_1, y_2, z]/I$*, where* I *is the ideal generated by the diagonal* 4×4 *minors of the matrix*

$$\begin{pmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ & 0 & y_1 & y_2 & z \\ & & 0 & -z & -A \\ & & & 0 & -B \\ -\mathrm{sym} & & & & 0 \end{pmatrix} \quad \text{of degrees} \quad \begin{pmatrix} -1 & 0 & 1 & 1 & 2 \\ & 1 & 2 & 2 & 3 \\ & & 3 & 3 & 4 \\ & & & 3 & 4 \\ \mathrm{sym} & & & & 4 \end{pmatrix},$$

and $R(X) = k[x_0, x_1, \dots, z]/I$*, where* I *is described similarly in terms of the matrix*

$$\begin{pmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ & 0 & y_1 + \cdots & y_2 & z \\ & & 0 & -z + \cdots & -A + \cdots \\ & & & 0 & -B + \cdots \\ & -\mathrm{sym} & & & 0 \end{pmatrix};$$

the $\cdots$ *correspond to adding an arbitrary multiple of* x_0 *to the matrix entry for* $R(C)$*.*

Proof (I) is very easy from standard facts on curves: C cannot be hyperelliptic, because a special free linear system of degree 4 would have to be $2g^1_2$, which contradicts the fact that $h^0(\mathcal{O}_C(K_{X|C})) = 2$. Thus the monomials x_1^2, x_1x_2, x_2^2 are linearly independent, and y_1, y_2 are a complementary basis. In its canonical embedding C is contained in a quadric of rank 3, the image of $\mathbb{P}(1, 1, 2, 2)$, the generators of which cut out the given g^1_4; from this, C cannot be trigonal, so the two other quadrics through C provide the relations f_4, g_4.

(II) C is a nonsingular curve of genus 5 with a g^1_3 and a point P such that $P + g^1_3 \in \frac{1}{2}K_C$. It follows from RR that $|2P + g^1_3|$ maps C birationally to a plane quintic with a cusp, with the g^1_3 corresponding to lines in the plane through the cusp. The canonical map of C is obtained by blowing up $\mathbb{P}^2$ at the cusp point, then embedding it as the cubic scroll $\mathbb{F}_1 \subset \mathbb{P}^4$.

Writing out the ring $R(C)$ is a valuable exercise for the reader who wants to learn how to calculate these types of graded rings (that is, how to get algebra out of the geometry). Write $u\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P)$ for the trivial inclusion and t_1, t_2 for a basis of the g^1_3, chosen so that $t_1(P) = 0$; then $x_1 = ut_1$, $x_2 = ut_2$ is the basis of $H^0(C, P + g^1_3)$. Let y_1, y_2 be a complementary basis of $|K_C| = |2K_{X|C}|$ chosen to vanish on a positive section of $\mathbb{F}_1$, and such that $(y_1 : y_2) = (t_1 : t_2)$. Choose z to be a complementary basis element of $|3K_{X|C}| = |K_C + g^1_3 + P|$ not vanishing at P. It is now easy to write out the ideal of relations holding between these generators, and to manipulate them into the Pfaffian format of the statement.

The results for $R(X)$ follow from those for $R(C)$ using the hyperplane section principle [R2], (1.2) and the structure theorem for codimension 3 Gorenstein rings (or by a deformation calculation similar to that of Section 3).

Applications 2.2 *(a) A surface S of type (II) has a -1-elliptic cycle E so that $|K_S + E|\colon S \to \overline{S}_5 \subset \mathbb{P}^3$ is birational to a quintic with an elliptic Gorenstein singularity of degree 1 (a singularity of type $x^2 + y^3 + z^6 + \cdots$).*

(b) A surface of type (II) has a small deformation of type (I).

Proof (a) E is obtained by setting to zero the top row of the matrix defining $R(X)$, that is, $x_1 = x_2 = y_1$; clearly this is a hypersurface in the weighted projective space $\mathbb{P}(1, 2, 3)$ corresponding to x_0, y_2 and z with defining equation $z^2 + \cdots$

Conversely, it is fun to write out the canonical ring of the resolution of a quintic $\overline{S}_5 \subset \mathbb{P}^3$ with an elliptic Gorenstein singularity of degree 1 and to recover the Pfaffian format of Theorem 2.1.

(b) Making an arbitrary small change of the entries of the matrix defining X leads to a flat deformation (because the syzygies are all implied by the Pfaffian format). In particular, I can fill in the $(1, 2)$th entry of the matrix with a value $\lambda \neq 0$; it is easy to see that then the first 4×4 Pfaffian becomes $\lambda \cdot (-z) - x_1 y_2 + x_2 y_1$. Thus z is expressible in the new ring $R(X_\lambda)$, as a polynomial in the other variables, which means the new surface X_λ is of type (I). Q.E.D.

3 Algebra: harder cases, deformation theory

In this case the rings are more complicated, and the study of the ring of the surface $R(X)$ makes substantial use of the curve case and the deformation theory of [R2].

Theorem 3.1 *(III) $R(C, K_{X|C}) = k[x_1, x_2, y_1, y_2, z_1, z_2]/I$; the ideal I is generated by 9 relations*

$$\operatorname{rank} A \leq 1 \qquad \textit{and} \qquad AM({}^tA) = 0,$$

where

$$A = \begin{pmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{pmatrix} \quad \textit{and} \quad M = \begin{pmatrix} h & & & \\ & y_1 & & \\ & & y_2 & \\ & & & -1 \end{pmatrix}.$$

(III$_a$) $R(C)$ has the same description, with the matrixes

$$A = \begin{pmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{pmatrix} \quad \textit{and} \quad M = \begin{pmatrix} h & & & \\ & 0 & & \\ & & \lambda y_1 + y_2 & \\ & & & -1 \end{pmatrix}.$$

Here h is some quartic in x_1, x_2, y_1, y_2. The condition $\operatorname{rank} A \le 1$ means that the minors of A vanish in $R(C)$, providing 6 of the generators of the ideal of relations I; $AM({}^tA) = 0$ is a set of 3 relations, for example

$$\begin{aligned} z_1^2 &= x_1^2 h + y_1^3 + x_2^4 y_2 \\ z_1 z_2 &= x_1 x_2 h + y_1^2 x_1^2 + x_2^2 y_2^2 \\ z_2^2 &= x_2^2 h + y_1 x_1^4 + y_2^3. \end{aligned}$$

Proof In case (III), C is a nonsingular hyperelliptic curve of genus 5, and the restriction of K_X is of the form

$$K_{X|C} = g_2^1 + P_1 + P_2,$$

where P_1, P_2 are the given base points. $2K_{X|C} = K_C = 4g_2^1$, and every effective divisor of $|K_C|$ is made up of 4 elements of the g_2^1, so that $2P_1 \sim 2P_2 \sim g_2^1$ and P_1, P_2 are Weierstrass points of C. There is a completely general and more-or-less automatic procedure [R2], Section 4 for writing out rings of this form over hyperelliptic curves, including the singular case needed for (III$_a$); I just give the flavour, by sketching where the generators and relations come from: the given divisor $D = g_2^1 + P_1 + P_2$ corresponds to a division of the 12 Weierstrass points of C into two groups, $\{P_1, P_2\}$ and the remainder $\{P_3, \dots, P_{12}\}$. Let $u\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1 + P_2)$ and $v\colon \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_3 + \cdots + P_{12})$ be the two inclusions, and t_1, t_2 a basis of the g_2^1 chosen so that $t_1(P_1) = t_2(P_2) = 0$. Then for $m = 1, 2, 3$ a basis of $H^0(C, mD)$ is found as follows:

- For $m = 1$: $x_1 = ut_1$, $x_2 = ut_2$.

- For $m = 2$: $x_1^2 = t_1^3 t_2$, $x_1 x_2 = t_1^2 t_2^2$, $x_2^2 = t_1 t_2^3$, so set $y_1 = t_1^4$, $y_2 = t_2^4$.
- For $m = 3$: $z_1 = vt_1$, $z_2 = vt_2$.

The relations $\operatorname{rank} A \leq 1$ are obvious monomial relations between the generators x_1, x_2, y_1, y_2 and z, asserting that the ratio $(t_1 : t_2)$ between top and bottom rows of A is well defined. The three final relations are derived from the single relation $v^2 = f_{10}(t_1, t_2)$, where f_{10} is the polynomial defining the 10 Weierstrass points $P_3, \ldots, P_{12}$. Q.E.D.

$R(X)$ is obtained from $R(C)$ by deformation theory. To be able to do calculate the deformation groups it is essential to know the *syzygies* yoking the 9 relations of Theorem 3.1; if the relations are R_i then the syzygies are by definition the identities $\sum L_{ji} R_i$ holding between them in the polynomial ring.

Proposition 3.2 *There are* 16 *syzygies that hold between the* 9 *relations*

$$\operatorname{rank} A \leq 1 \quad \textit{and} \quad AM({}^tA) = 0$$

of Theorem 3.1. They are obtained by the two following standard tricks, each of which leads to 8 *syzygies:*

(i) Take a 3×3 *minor of a matrix obtained by repeating one of the two rows of* A*; this is identically zero, but it is also a linear combination of* 3 *of the* 2×2 *minors of* A*.*

(ii) Write $N = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$*; then* ${}^tANA$ *is a* 4×4 *skew matrix whose* 6 *entries are the* 2×2 *minors of* A*. Now in the identity*

$${}^tAN(AM({}^tA)) = ({}^tANA)M({}^tA),$$

the left-hand side is ${}^tAN$ *times the final* 3 *relations* $AM({}^tA)$*, whereas the right-hand side is the first* 6 *relations times* $M({}^tA)$*.*

3.1 Deformation theory

Here is a brief description of the material of [R2], Section 1, which forms the background to the proof of the main theorem. Given a graded ring $R^{(0)} = R(C, K_{X|C})$, together with a description of it by generators, relations and syzygies, consider the set of rings R with a fixed element $x_0 \in R_1$ (that is, x_0 is homogeneous of degree 1) such that x_0 is a non-zerodivisor of R and $R^{(0)} = R/(x_0)$; the canonical ring $R(X)$ is of this form, and to find X is essentially the same as to find a ring $R = R(X)$ solving the above algebraic problem, together with the requirement that $X = \operatorname{Proj} R(X)$ is a canonical surface (that is, has no worse than Du Val singularities).

The infinitesimal view of this problem is to define and study rings $R^{(n)}$ as nth order infinitesimal extensions of $R^{(0)}$. If R is known then $R^{(n)} = R/(x_0^{n+1})$, and $R^{(n)}$ fits into an extension sequence

$$R^{(0)} \cong (x_0^n) \hookrightarrow R^{(n)} \to R^{(n-1)}.$$

Consider the problem of recovering $R^{(n)}$ in terms of $R^{(0)}$ and $R^{(n-1)}$. To reduce this to a calculation, I have to (i) take all the relations modulo x^n defining $R^{(n-1)}$; (ii) note down all the syzygies yoking them; then (iii)

$$\{R^{(n)}\} = \left\{ \begin{matrix} \text{ways of extending relns modulo } x_0^{n+1} \\ \text{preserving the syzygies} \end{matrix} \right\}.$$

The main result of deformation theory [R2], Section 1 is that this is an affine linear problem; that is, (a) assuming one solution $R^{(n)}$ exists then there is a vector space T^1_{-n} of solutions; (b) there is an obstruction $\operatorname{obs}(R^{(n-1)})$ which lives in a vector space T^2_{-n}, with $\operatorname{obs}(R^{(n-1)}) = 0$ a necessary and sufficient for one solution $R^{(n)}$ to exist; (c) the obstruction can be made to depend in a bilinear way on $R^{(n-1)}$ and the "normal data" of $R^{(n-1)}$ (that is, the way in which the syzygies of $R^{(0)}$ have been lifted to syzygies for $R^{(n-1)}$). Note that the vector spaces T^1_{-n} and T^2_{-n} depend only on the initial $R^{(0)}$ and the degree n of x_0^n, and not on the choice of $R^{(n-1)}$.

3.2 Rolling factors

Theorem 3.1 was stated in terms of the matrix format $AM({}^tA) = 0$ with a symmetric matrix M, but there is another way of saying the result. Namely, the first 6 relations can be written as a 2×4 determinantal rank $A \le 1$, and the last 3 in the form $z_1^2 = P_1$, $z_1 z_2 = P_2$, $z_2^2 = P_3$ where P_1, P_2, P_3 are obtained from one another by *rolling factors*: that is, P_1 is a sum of terms each of which has a factor that is an entry a_{1i} of the first row of A, and $P_1 \mapsto P_2$ consists of replacing one factor a_{1i} in each term of P_1 with the corresponding entry a_{2i} from the second row (in Theorem 3.1 the entries of A are all monomials); $P_2 \mapsto P_3$ is the same procedure. It is clear that this format automatically gives rise to certain syzygies. In general, the rolling factors format is more general than the $AM({}^tA) = 0$ format, and allows me to describe some obstructed deformations. (See Section 5 for more discussion.)

This is the main result:

Theorem 3.3 *Let X be a surface in case (III) or (III$_a$). Then $R(X) = R(X, K_X) = k[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/I$, where the ideal I is generated by 9 relations in the rolling factors format described above. In detail:*

Case (III) *Set*

$$X_1 = x_1^2 + a_2x_0x_1 + e_1x_0^2, \qquad Y_1 = y_1 + d_1x_0x_1 + i_1x_0^2,$$
$$\text{and}$$
$$X_2 = x_2^2 - b_4x_0x_2 - f_2x_0^2, \qquad Y_2 = y_2 + d_2x_0x_2 + i_2x_0^2,$$

and write $A = \begin{pmatrix} x_1 & y_1 & X_2 & z_1 \\ x_2 & X_1 & y_2 & z_2 \end{pmatrix}$. *Then the first 6 relations are given by* $\operatorname{rank} A \le 1$, *and the last 3 by*

$$\left.\begin{array}{rcllll} z_1^2 &=& x_1^2H & + y_1^2Y_1 & + X_2^2Y_2 \\ z_1z_2 &=& x_1x_2H & + y_1X_1Y_1 & + y_2X_2Y_2 \\ z_2^2 &=& x_2^2H & + X_1^2Y_1 & + y_2^2Y_2 \end{array}\right\} +$$
$$+ x_0^3 \left\{\begin{array}{ll} +2l_1x_1y_1 & + 2l_2x_1X_2 \\ +l_1(x_1X_1 + x_2y_1) & + l_2(x_1y_2 + x_2X_2) \\ +2l_1x_2X_1 & + 2l_2x_2y_2 \end{array}\right\} +$$
$$+ x_0^4 \left\{\begin{array}{l} +n_1y_1 + n_3x_1x_2 - n_3b_4x_0x_1, \\ +n_1X_1 + n_3x_2^2 - n_3b_4x_0x_2 \\ \qquad (= n_1x_1^2 + n_3X_2 + n_1a_2x_0x_1), \\ +n_1x_1x_2 + n_3y_2 + n_1a_2x_0x_2. \end{array}\right.$$

Here $H = h + x_0h' + \cdots$ is a quartic, and the undefined symbols a_2, e_1 etc. are just constants in k that can be chosen freely, except that (wake up, this is important!) $n_1e_1 + n_3f_2 = 0$ *must hold; plugging in the definition of X_1, X_2, one sees that this is equivalent to the bracketed equality in the last line of the display.*

Case (III$_a$) *The same description, with* $A = \begin{pmatrix} x_1 & X_1 & y_1 & z_1 \\ x_2 & X_2 & y_2 & z_2 \end{pmatrix}$, *where*

$$\begin{aligned} X_1 &= x_2^2 - a_3x_0x_2 - e_2x_0^2, \\ X_2 &= y_1 + a_1x_0x_1 + e_1x_0^2, \\ Y &= \lambda y_1 + y_2 + x_0(d_1x_1 + d_2x_2) + i_2x_0^2, \end{aligned}$$

and the last 3 *relations have the form*

$$\left.\begin{array}{rcllll} z_1^2 &=& x_1^2H & +Yy_1^2 & +2i_1X_1y_1 \\ z_1z_2 &=& x_1x_2H & +Yy_1y_2 & +i_1(x_1y_2+X_2y_1) \\ z_2^2 &=& x_2^2H & +Yy_2^2 & +2i_1X_2y_2 \end{array}\right\}+$$

$$+\,x_0^3\left\{\begin{array}{ll} +2l_1x_1X_1 & +2l_2x_1y_1 \\ +l_1(x_1X_2+x_2X_1) & +l_2(x_1y_2+x_2y_1) \\ +2l_1x_2X_2 & +2l_2x_2y_2 \end{array}\right\}+$$

$$+\,x_0^4\left\{\begin{array}{l} +n_2x_1x_2+n_3X_1-n_2a_3x_0x_1, \\ +n_2x_2^2+n_3X_2-n_2a_3x_0x_2 \\ \qquad (=n_2X_1+n_3y_1+n_3a_1x_0x_1), \\ +n_2X_2+n_3y_2+n_3a_1x_0x_2, \end{array}\right.$$

with the restriction $n_2e_2+n_3e_1=0$ *required to achieve the equality sticking out in the last line.*

Conversely, for any choice of the quartic H *and of the deformation variables* a_2, b_4, *etc. satisfying* $n_1e_1+n_3f_2=0$ *resp.* $n_2e_2+n_3e_1=0$, *the* 9 *relations given above define a ring* R *such that* x_0 *is a non-zerodivisor and* $R/(x_0)=R(C)$. *For a general choice,* $X=\operatorname{Proj} R$ *is a nonsingular surface in (III), and has a Du Val singularity* A_1 *at* $x_0=x_1=x_2=0$ *in* (III$_a$)*; if* $e_1=f_2=0$ *in (III) or* $e_1=e_2=0$ *in* (III$_a$) *then* X *is singular at* $P_0=(1,0,\dots)$.

3.3 Remarks

(i) In (III), if $n_1=n_3=0$ then the set of 9 relations can be put back in the $AM({}^tA)=0$ format, with M the matrix

$$\begin{pmatrix} H & l_1x_0^3 & l_2x_0^3 & \\ & Y_1 & & \\ & & Y_2 & \\ & \text{sym} & & -1 \end{pmatrix}.$$

This is definitely not possible if n_1 or $n_3\neq 0$.

Nevertheless, the groups T^1_{-n} can be computed for each n (I give a sample of this calculation below), and it happens that first order deformations in degrees -1, -2, -3 can be manipulated back into the determinantal format (that is, the determinantal format is *complete* in these degrees); this is valuable as a way of understanding the computation, and that is why the last 3 relations have been massaged as far as possible into quadratic expressions in the rows of A. As I noted in the above proposition, all the syzygies holding between the given set of 9 relations are implied by the determinantal format. This means that changing the entries of A and M by adding multiples of x_0^n automatically gives rise to flat infinitesimal extensions of $R(C)$, and that these extensions are unobstructed (that is, the determinantal format is *flexible*).

(ii) The requirement $n_1 e_1 + n_3 f_2 = 0$ is the single obstruction between the deformation variables e_i, f_i in degree -2 and n_1, n_3 in degree -4. Since it only affects the term in x_0^6, and occurs only at the very end of a long calculation, it is rather easy to miss the point.

(iii) Similarly for (III_a). It is probably possible to state and prove the theorem without dividing into cases, but it is not clear that it is worth the effort.

The bulk of the computation reduces to first order considerations. The ideal situation would be if the determinantal format was complete in each degree < 0: the relations for $R^{(n-1)}$ could then be written in the determinantal format, so that $R^{(n-1)}$ is unobstructed by flexibility; then any choice of $R^{(n)}$ differs from a standard determinantal extension by an element of T^1_{-n}, and by completeness this element could be obtained by varying the entries of the matrixes, so that in turn $R^{(n)}$ could be put in the determinantal form.

In degree -4 this fails, and in each of the two cases (III) and (III_a) there is a 2 dimensional family of deformations that cannot be fitted into the determinantal format. By this stage the computations are fairly small, and it can be shown that these nondeterminantal deformations are obstructed. The theorem follows from this.

3.4 The relations and syzygies for $R(C)$

It is easy to write out the 9 relations defining $R(C)$:

$$\begin{aligned}
R_1 &= x_1^3 - x_2y_1\\
R_2 &= x_1y_2 - x_2^3\\
R_3 &= y_1y_2 - x_1^2x_2^2\\
R_4 &= x_1z_2 - x_2z_1\\
R_5 &= y_1z_2 - x_1^2z_1\\
R_6 &= y_2z_1 - x_2^2z_2\\
R_7 &= -z_1^2 + x_1^2h + y_1^3 + x_2^4y_2\\
R_8 &= -z_1z_2 + x_1x_2h + x_1^2y_1^2 + x_2^2y_2^2\\
R_9 &= -z_2^2 + x_2^2h + x_1^4y_1 + y_2^3.
\end{aligned}$$

I only need the following set of 5 syzygies, because it is easy to check that every other syzygy has a monomial multiple that is a linear combination of these.

$$\begin{aligned}
S_1\colon \quad x_1R_3 &\equiv y_1R_2 - x_2^2R_1\\
S_2\colon \quad x_1R_5 &\equiv y_1R_4 - z_1R_1\\
S_3\colon \quad x_1R_6 &\equiv x_2^2R_4 - z_1R_2\\
S_4\colon \quad x_1R_8 &\equiv x_2R_7 - z_1R_4 + y_1^2R_1 + x_2^2y_2R_2\\
S_5\colon \quad x_1R_9 &\equiv x_2R_8 - z_2R_4 + y_1x_1^2R_1 + y_2^2R_2.
\end{aligned}$$

The calculation of the first order deformation space in degree $-n$ proceeds as follows: let ξ be an indeterminate weighted with degree n. I write down the relations modulo ξ^2 as $R_i + \xi R_i'$, where $R_i' \in R(C)$ is a general element homogeneous of degree $\deg R_i - n$. Then each syzygy implies an *equality* in $R(C)$, giving linear conditions on the R_i'.

This calculation really must be done by computer algebra, since in my experience, working by hand one inevitably cuts corners and makes errors; having the calculation down in a computer file makes it into a repeatable experiment, enabling one to concentrate on the key logical steps, rather than having to spend time on the mechanical processes of polynomial multiplication. Also, in hand calculations, one often has to pass to a normal form to reduce the number of variables before knowing the shape of the final result. (I will send on request a Maple file with the complete calculations of the proof of Theorem 3.3; this is a computer-assisted hand calculation rather than a genuine implementation of the algorithm of [R2], Section 6.)

3.5 Sample calculation

In degree -1, write out R'_1, R'_2 as quadratics in x_1, x_2, y_1, y_2 with general coefficients:

$$\begin{aligned} R'_1 &= a_1y_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_2^2 + a_5y_2; \\ R'_2 &= b_1y_1 + b_2x_1^2 + b_3x_1x_2 + b_4x_2^2 + b_5y_2. \end{aligned}$$

The equality arising from the first syzygy says that $y_1R'_2 - x_2^2R'_1$ is divisible by x_1 in $R(C)$; multiplying this out explicitly, one sees that the two monomials $b_1y_1^2$ and $a_5x_2^2y_2$ are linearly independent of the multiples of x_1 in $H^0(C, \mathcal{O}_C(4))$, hence $b_1 = a_5 = 0$. Carrying out the division gives the value of R'_3 below.

Next write out R'_4 as a general cubic:

$$R'_4 = c_1x_1y_1 + c_2x_1^3 + c_3x_1^2x_2 + c_4x_1x_2^2 + c_5x_2^3 + c_6x_2y_2 + c_7z_1 + c_8z_2.$$

Then the monomial $(c_7 - a_1)y_1z_1$ is an obstruction to the divisibility of $y_1R'_4 - z_1R'_1$ by x_1, and this implies that $c_7 = a_1$; in exactly the same way, the monomial $(-c_8 + b_5)y_2z_1$ obstructs the divisibility of $x_2^2R'_4 - z_1R'_2$, so that $c_8 = b_5$. Carrying out the divisions gives the values of R'_5 and R'_6. The story so far:

$$\begin{aligned} R'_1 &= a_1y_1 + a_2x_1^2 + a_3x_1x_2 + a_4x_2^2 \\ R'_2 &= b_2x_1^2 + b_3x_1x_2 + b_4x_2^2 + b_5y_2 \\ R'_3 &= b_2x_1y_1 + b_3x_1^3 + (b_4 - a_1)x_1^2x_2 + (b_5 - a_2)x_1x_2^2 \\ &\qquad -a_3x_2^3 - a_4x_2y_2 \\ R'_4 &= c_1x_1y_1 + c_2x_1^3 + \cdots + c_6x_2y_2 + a_1z_1 + b_5z_2 \\ R'_5 &= c_1y_1^2 + c_2y_1x_1^2 + \cdots + c_6x_1x_2^3 \\ &\qquad +(b_5 - a_2)x_1z_1 - a_3x_2z_1 - a_4x_2z_2 \\ R'_6 &= -c_1x_1^3x_2 - \cdots - c_6y_2^2 \\ &\qquad +b_2x_1z_1 + b_3x_2z_1 + (-a_1 + b_4)x_2z_2. \end{aligned}$$

Now it is not hard to see that these can be squeezed into the matrix form $\operatorname{rank}(A + x_0A') \leq 1$, where A is as in Theorem 3.1 and

$$A' = \begin{pmatrix} b_5 & -a_3x_1 - a_4x_2 & -b_4x_2 & -c_5x_2^2 - c_6y_2 \\ -a_1 & (a_2 - b_5)x_1 & b_2x_1 + b_3x_2 & c_1y_1 + \cdots + c_4x_2^2 \end{pmatrix}.$$

Choosing new coordinates $x_1 + b_5x_0$, $x_2 - a_1x_0$, $y_1 - a_3x_0x_1 - a_4x_0x_2$, $y_2 + b_2x_0x_1 + b_3x_0x_2$, $z_1 - c_5x_0x_2^2 - c_6x_0y_2$ and $z_2 + x_0c_1y_1 + \cdots + c_4x_2^2$ sets $a_1 = a_3 = a_4 = b_2 = b_3 = b_5 = c_1 = \cdots = c_6 = 0$, and reduces the 6 equations to

$$\operatorname{rank}\begin{pmatrix} x_1 & y_1 & x_2^2 - b_4x_0x_2 & z_1 \\ x_2 & x_1^2 + a_2x_0x_1 & y_2 & z_2 \end{pmatrix} \le 1.$$

Similarly, the last 3 equations of the R'_i are

$$\begin{array}{rclclclcl} R'_7 &=& x_1^2h' &+& d_1x_1x_2^2y_2 & & &-& 2b_4x_2^3y_2 \\ R'_8 &=& x_1x_2h' &+& d_1x_2^3y_2 &+& a_2x_1y_1^2 &-& b_4x_2y_2^2 \\ R'_9 &=& x_2^2h' &+& d_1x_2y_2^2 &+& 2a_2x_1^3y_1, & & \end{array}$$

(for clarity I am omitting terms in z_1 and z_2 that can be killed by completing the square followed by column operations on A), and it is clear how to squeeze these into the determinantal form.

In degree -2 and -3 the computation is exactly similar. In degree -2, the first 6 equations must be modified to

$$\operatorname{rank}\begin{pmatrix} x_1 & y_1 & x_2^2 - f_2\xi & z_1 \\ x_2 & x_1^2 + e_1\xi & y_2 & z_2 \end{pmatrix} \le 1,$$

where $\deg\xi = 2$. The two deformation variables e_1 and f_2 are important: $(e_1, f_2) \ne (0,0)$ is the condition that the surface X in the weighted projective space $\mathbb{P}(1^3, 2^2, 3^2) = \operatorname{Proj} k[x_0, \ldots, z_2]$ does not pass through the point $P_0 = (1, 0, \ldots)$.

3.6 The obstructed deformations in degree -4

In degree -4 the first order computation is very easy: $R'_i = 0$ for $i = 1, \ldots, 6$ for reasons of degree, and

$$\begin{array}{rclclcl} R'_7 &=& n_1y_1 &+& n_2x_1^2 &+& n_3x_1x_2 \\ R'_8 &=& n_1x_1^2 &+& n_2x_1x_2 &+& n_3x_2^2 \\ R'_9 &=& n_1x_1x_2 &+& n_2x_2^2 &+& n_3y_2. \end{array}$$

Here the n_2 terms can be easily accomodated in the determinantal format (in the same way as h' in degree -1), but the n_1 and n_3 terms can certainly not: because altering M adds in to R'_7, R'_8, R'_9 quadratic terms in the rows of A, and you cannot possibly hit y_1, y_2 this way.

Let η be the deformation variable with $\deg \eta = 4$; consider the deformation $R^{(\xi,\eta)}$ of $R(C)$ over $k[\xi,\eta]/(\xi,\eta)^2$ defined by the two preceding displays.

Claim 3.4 $R^{(\xi,\eta)}$ *can be extended to a deformation over* $k[\xi,\eta]/(\xi^2,\eta^2)$ *if and only if* $e_1n_1 + f_2n_3 = 0$.

The problem is to fix up the deformation terms multiplying $\xi\eta$ so that the syzygies extend (compare [R2], (5.15–16) for a similar calculation). Here ξ plays the role of x_0^2 and η that of x_0^4, so the claim determines the obstruction to lifting the ring $R^{(4)}$ to a ring $R^{(6)}$ by fixing up the terms multiplying x_0^6. Nothing of much interest happens in degree -5 and degrees < -6, so that this is the essential point in the proof of Theorem 3.3.

Proof of claim In temporary notation, write $R_i + \xi R'_i + \eta R''_i$ for the deformed R_i (the equations defining $R^{(\xi,\eta)}$). In each of the calculations in degree -2 and -4, I have used the syzygy S_4 to give equalities in $R(C)$; it becomes an identity again (over $k[\xi,\eta]/(\xi,\eta)^2$) on adding in certain multiples of the R_i (the "credit card charge" for using the relations). Thus

$$\begin{aligned}
&S_4 + \xi S'_4 + \eta S''_4 : \\
&\quad x_1(R_8 + \xi R'_8 + \eta R''_8) \equiv \; x_2(R_7 + \xi R'_7 + \eta R''_7) - z_1(R_4 + \xi R'_4) \\
&\qquad\qquad\qquad\qquad\qquad + y_1^2(R_1 + \xi R'_1) + x_2^2 y_2(R_2 + \xi R'_2) \\
&\qquad\qquad\qquad\qquad\qquad + i_1\xi y_1 R_1 - f_2\xi y_2 R_2 + n_1\eta R_1.
\end{aligned}$$

Here I am omitting terms like $\eta R''_4$ which are zero, and in the last line, I have ignored the term $\xi R'_1$ *because I am working modulo* $\xi\eta$. Now to lift the ring to $k[\xi,\eta]/(\xi^2,\eta^2)$, I should take care of this last term, and I must adjust $R_i \mapsto R_i + \xi\eta R'''_i$ so as to arrange that the $\xi\eta$ terms of this syzygy is zero in $R(C)$. That is, as in the first order computations, I have to solve

$$x_1 R'''_8 = x_2 R'''_7 + n_1 R'_1$$

with $\deg R'''_7 = \deg R'''_8 = 0$. Looking up the value $R'_1 = e_1x_1$, this gives $R'''_8 = n_1e_1$ and $R'''_7 = 0$.

An identical computation with S_5 gives $R'''_8 = -n_3f_2$. This proves the claim.

Now to complete the proof of the theorem in Case (III): the deformation calculation sketched above shows that the relations defining R must be of the form given in the theorem; to show that these equations actually define a ring R with the required property, I have to show that the syzygies extend to all orders. Intuitively, this follows for reasons explained before the theorem, and in 5.3 I discuss another "format" due to Dicks that gives another proof. The fact that equations of this form define in general a nonsingular surface with $p_g = 3$, $K^2 = 4$ is best understood by making the link with Horikawa's geometric description of the surface as a double cover, for which see Section 4.

Near $P_0 = (1, 0, \dots)$, the weighted projective space $\mathbb{P}(1^3, 2^2, 3^2)$ is nonsingular and 6 dimensional, with local coordinates x_i/x_0, y_i/x_0^2, z_i/x_0^3. It is easy to see that if $e_1 = f_2 = 0$ then the first 6 equations all have multiplicity ≥ 2 at P_0; the final 3 relations can only cut down the dimension of the tangent space by 1 each, hence $\dim T_P X \geq 3$.

4 Geometric applications, moduli spaces

4.1 Curves

The idea of treating the curve problem systematically as a prelude to the study of surfaces was introduced by Ed Griffin [G]. Consider the classification of curves $(C, \mathcal{O}_C(1))$ of genus 5 with a halfcanonical polarisation (that is $K_C = \mathcal{O}_C(2)$) such that $h^0(\mathcal{O}_C(1)) = 2$. These divide into families (I), (II), (III) as in Section 1, Theorem 1; write $\mathcal{C}_{\mathrm{I}}$, $\mathcal{C}_{\mathrm{II}}$, $\mathcal{C}_{\mathrm{III}}$, for the corresponding moduli spaces (or their closures). The result on curves in Theorem 2.1 and Application (b) shows at once that $\mathcal{C}_{\mathrm{II}}$ is generically a smooth divisor in $\mathcal{C}_{\mathrm{I}}$. Dicks' result here is the following:

Theorem 4.1 *$\mathcal{C}_{\mathrm{I}}$ is nonsingular at a curve $C \in \mathcal{C}_{\mathrm{III}}$, and $\mathcal{C}_{\mathrm{III}}$ is smooth of codimension 2 in it. Moreover, $\mathcal{C}_{\mathrm{II}}$ has $\mathcal{C}_{\mathrm{III}}$ as an ordinary double locus; in other words, keeping P_1 or P_2 as base points are independent codimension 1 conditions on $(C, \mathcal{O}_C(1))$ in a neighbourhood of $C \in \mathcal{C}_{\mathrm{III}}$.*

Sketch Proof The following set of equations defines a deformation of a curve in $\mathcal{C}_{\mathrm{III}}$ depending on 2 parameters transverse to $\mathcal{C}_{\mathrm{III}}$ (the deformations inside $\mathcal{C}_{\text{III}}$ are obtained by changing the coefficients of the quartic h):

$$\begin{array}{lllllll}
R_1 & = & x_1^3 & -x_2y_1 & +rz_1 & & -s^2x_1y_2 \\
R_2 & = & x_1y_2 & -x_2^3 & +sz_2 & & +r^2x_2y_1 \\
R_3 & = & y_1y_2 & -x_1^2x_2^2 & +sx_1z_1 - rx_2z_2 & & +rsh \\
R_4 & = & x_1z_2 & -x_2z_1 & +ry_1^2 + sy_2^2 & & \\
R_5 & = & y_1z_2 & -x_1^2z_1 & -rx_2y_2^2 + sx_1x_2^2y_2 - rx_1h & & \\
R_6 & = & y_2z_1 & -x_2^2z_2 & +sx_1y_1^2 - rx_1^2x_2y_1 + sx_2h & & \\
R_7 & = & -z_1^2 & +x_1^2h & +y_1^3 & +x_2^4y_2 & -s^2y_2h - r^2x_2^2y_1y_2 \\
R_8 & = & -z_1z_2 & +x_1x_2h & +x_1^2y_1^2 & +x_2^2y_2^2 & -rsx_1x_2y_1y_2 \\
R_9 & = & -z_2^2 & +x_2^2h & +x_1^4y_1 & +y_2^3 & -r^2y_1h - s^2x_1^2y_1y_2.
\end{array}$$

This proves the theorem, since for a fixed value of (r, s), clearly z_1 or z_2 is in the subring generated by x_1, x_2, y_1, y_2 if and only if r or $s \neq 0$.

The linear terms in r, s can be derived by the same first order calculation as in Section 3, although it is somewhat tricky to get the normal form. Having chosen the linear terms, the quadratic terms are forced by the second order deformation calculation as in Section 3. (The derivation of the equations is not needed for the proof.)

To show that these equations define a flat deformation, one has to check that the syzygies $S_1, \dots, S_5$ of Section 3 extend, which is a long mechanical calculation. Alternatively, one knows how to calculate the space T_0^2 in which the obstructions live, and (although I have not done this) I bet it is zero. Q.E.D.

Dicks' mysterious derivation of these equations from the 4×4 diagonal Pfaffians of a certain 6×6 skew matrix is discussed in 5.3.

4.2 Applications to a single surface

With an appropriate amount of work, almost all the geometric properties of a surface S in (III) or (III$_a$) can be recovered from the algebra. I use the notation of Theorem 3.3. First of all, it is easy to see that S has a genus 2 pencil $|F|$ cutting out the g_2^1 on each general curve $C \in |K_S|$: fix the ratio $(\alpha : \beta)$ between top and bottom row of A, and the last 3 equations reduce to a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1^2, 3)$ with coordinates x_0, x_1, z_1 (if $\alpha \neq 0$). This must be a nonsingular curve of genus 2 for general $(\alpha : \beta)$ since S is of general type. Next, the reducible fibres of $|F|$ are manifest: in case (III), the fibre $\beta = 0$ has $x_2 = X_1 = y_2 = z_2$, and is a complete intersection $F_{2,6} \subset \mathbb{P}(1^2, 2, 3)$ (with coordinates (x_0, x_1, y_1, z_1)) defined by $0 = X_1 = x_1^2 + a_2 x_0 x_1 + e_1 x_0^2$ and a sextic coming from the last 3 relations. This is obviously reducible (or nonreduced if X_1 is a perfect square), and cannot be 2-connected.

Next, the structure of the 1-canonical map $\varphi\colon S \dashrightarrow \mathbb{P}^2$ can be understood in terms of eliminating the variables y_1, y_2 from $R(S, K_S)$. In case (III), the first two relations are $x_2 y_1 = x_1 X_1$, $x_1 y_2 = x_2 X_2$, so substitute $y_1 = x_1 X_1 / x_2$, $y_2 = x_2 X_2 / x_1$ in the equation for z_1^2, multiply through by $x_1^2 x_2^4$ to clear denominators, and Hey Presto! the defining equation of a double cover, in the form

$$\zeta^2 = x_1 x_2 \big(x_1^3 x_2^3 H + x_1^4 X_1^3 + x_2^4 X_2^3 + \cdots \big),$$

where $\zeta = x_1 x_2^2 z_1$. It is elementary to see that the right-hand side defines the two axes $x_1 x_2 = 0$ together with a plane 10-ic having two pairs of triple points at $X_1 = x_2 = 0$, and $X_2 = x_1 = 0$ (infinitely near if X_1 resp. X_2 is a perfect square) and a 4-ple point at $(1, 0, 0)$. Of course, the genus 2 pencil of the surface corresponds to the ratio (x_1, x_2), that is, to lines through $(1, 0, 0)$. In general $\varphi^{-1}(1, 0, 0) = E$ is a nonsingular elliptic curve with $E^2 = -2$, $EF = 2$, but E can split off a -2-curve, giving rise to the $e_1 = f_2 = 0$ singularity referred to at the end of Theorem 4.

4.3 Applications to moduli spaces

Write $\mathcal{S}_{\mathrm{I}}$, $\mathcal{S}_{\mathrm{II}}$, $\mathcal{S}_{\mathrm{III}}$ and $\mathcal{S}_{\mathrm{III_a}}$ for the moduli spaces of surfaces in the 4 cases. The Horikawa diagram is

$$\begin{array}{ccccc} & & \mathrm{I} & & \\ & \swarrow & & \searrow & \\ \mathrm{II} & & & & \mathrm{III} \\ & & & & & \searrow & \\ & & & & & & \mathrm{III}_a; \end{array}$$

each of the oblique arrows means an inclusion between the moduli spaces. I believe that each is generically an inclusion of a smooth divisor. The nontrivial case of this that remains to be proved is $\mathrm{I} \to \mathrm{III}$; this can be handled by the same kind of methods: it must be possible to write down equations similar to those of Theorem 4.1, and Dicks claims to do this, although I have not had time to study his long calculations in detail (there are at least some minor errors).

It is interesting to compare our algebraic methods with those of Horikawa [H1], [H2]; he deduces the existence of the oblique arrows essentially by the logical process of elimination: he knows the dimension of $\mathcal{S}_{\mathrm{III}}$ near a general surface S by studying the model as a double plane, and Kodaira–Spencer deformation theory says that the local deformation space of S has bigger dimension. Thus his proof depends on hard analysis, rather than our long but elementary polynomial calculations. However, the analysis also contains obstruction calculations, and in good cases, these will reduce to polynomial calculations by finite determinacy considerations.

Note that the general surface in (III) does not have a small deformation to (II); in other words, in contrast to the curve case, the two base points of $|K|$ are linked, and you cannot get rid of one without the other.

4.4 Problem

We still do not know whether *special* surfaces in (III) can deform to (II); for example, what about those with $e_1 = f_2 = 0$?

5 Speculation: Gorenstein in small codimension

There are structure theorems for Cohen–Macaulay rings of codimension 2 and Gorenstein rings of codimension 3; the famous theorem of Buchsbaum and Eisenbud says that a codimension 3 Gorenstein variety is defined by the $2k \times 2k$ diagonal Pfaffians of a $(2k+1) \times (2k+1)$ skew matrix. In codimension one higher, the commutative algebra literature is quite extensive, but does not seem to get anywhere (or at least, not anywhere I want to go). It seems to be known that a codimension 4 Gorenstein variety either has an odd number ≥ 7 of defining equations, or is a Cartier divisor in a codimension 3 Gorenstein variety. The simplest case, due to Kustin and Miller has 7 equations in the linear algebra format

$$A\mathbf{x} = 0, \qquad t\mathbf{x} = \bigwedge^3 A$$

where $\mathbf{x}$ is a 1×4 column vectors, A a 3×3 matrix, and t a scalar; if all the entries are general forms on $\mathbb{P}^6$ of the smallest degrees that make sense then the equations define a canonically embedded surface with $p_g = 7$, $K^2 = 17$, that is, degree 1 more than the complete intersection of 4 quadrics.

5.1 Coindex

If $X, \mathcal{O}_X(1)$ is a projectively Gorenstein polarised variety, its *coindex* is defined to be $k + 1 + \dim X$, where k is such that $K_X = \mathcal{O}_X(k)$; thus $\mathbb{P}^n$ has coindex 0, the quadric $Q \subset \mathbb{P}^{n+1}$ coindex 1, and an elliptic curve, del Pezzo surface, Fano 3-fold of index 2 etc. coindex 2. A similar definition is possible for a local ring (say normal Gorenstein over a field of characteristic zero) in terms of the smallest k such that $m^k \cdot \omega_X \subset f_*\omega_Y$ where $f\colon Y \to X$ is a resolution, so that a nonsingular point has coindex 0, a Du Val surface singularity or higher dimensional cDV point coindex 1, an elliptic Gorenstein surface singularity or general 3-fold rational Gorenstein point coindex 2, and a cone over a canonical curve or a weighted cone over a K3 with Du Val singularities coindex 3, and so on. The argument of [YPG], (3.10) shows that the coindex can only go down on taking a general hyperplane section. For a Gorenstein local Artinian ring (A, m), the coindex is by definition the smallest k with $m^{k+1} = 0$, so that e.g., coindex = 3 means

$$\begin{array}{cc} \circ & A/m \\ \circ \cdots \circ & m/m^2 \\ \circ \cdots \circ & m^2/m^3 \\ \circ & m^3/m^4 \end{array}$$

with A/m dual to m^3/m^4 and m/m^2 dual to m^2/m^3. For a Gorenstein curve singularity (or a numerical semigroup algebra), the coindex is the smallest k such that m^k is contained in the conductor ideal.

My experience that the commonly occuring Gorenstein varieties of codimension 4 or 5 often fit into a limited number of patterns is based mainly on studying 3-fold canonical singularities, K3s, canonical surfaces and 3-folds etc; that is, rings of small coindex. Because I am mainly interested in geometry, I usually work, at least implicitly, with a bound on the coindex; this is a condition not in common use among commutative algebraists. The coindex certainly imposes restrictions on the format: for example, a codimension 3 Gorenstein ring is defined by the Pfaffians of a $(2k + 1) \times (2k + 1)$ skew matrix P; assuming without loss of generality that every entry of P is in the maximal ideal n of the ambient space, the $2k + 1$ defining equations are in n^k. If the entries of P are generic linear forms then from the free resolution

$$0 \to \mathcal{O}(-2k-1) \to (2k+1)\mathcal{O}(-k-1) \to (2k+1)\mathcal{O}(-k) \to \mathcal{O} \to 0$$

it follows that the coindex is $2k - 2$; presumably in any case the coindex is $\geq 2k - 2$, which means, for example, that a codimension 3 weighted cone over a K3 with Du Val singularities must be either the complete intersection $X_{2,2,2} \subset \mathbb{P}^5$ or a 5×5 Pfaffian. Thus it is probable that if there is a good structure theory for Gorenstein in codimension 4, only the simpler formats will be important for the kind of geometric applications I have in mind.

5.2 Some favourite formats

A *format* is a way of writing down a set of equations defining a variety or singularity, depending on certain entries; I do not really know a proper definition. A format is only useful if it predicts all the syzygies yoking the defining equations. For an example, see the proposition in Section 3; in that case the format was *flexible*, since arbitrary (small) changes in the entries of the matrixes A and M are allowed, and the same set of syzygies hold. There is a closely related more general format due to Dicks, which however is not flexible: take a 2×4 matrix A and a 4×2 matrix Y satisfying the requirement that the product AY is a symmetric 2×2 matrix; then $\operatorname{rank} A \leq 1$ and $AY = 0$ is a set of 9 relations defining a codimension 4 Gorenstein variety, and the 16 syzygies between them are essentially the same as in the proposition in Section 3. This includes the $AM({}^tA)$ format as the special case $Y = M({}^tA)$; the equations of Theorem 3.3 can be fitted into Dicks' format: the curious equality in the last line of the displays is exactly what is needed for this. This format is inflexible, since $(AY)_{12} = (AY)_{21}$ is a nontrivial set of conditions on the entries of A and Y; thus it describes in general certain obstructed deformations.

The rolling factors format of Section 3 occurs very often in connection with divisors in scrolls. According to Corrado Segre and Pasquale del Pezzo (in the 1880s), the equations defining the scroll $F = \operatorname{Proj}_{\mathbb{P}^1}(\mathcal{O}(a, b, \dots))$ can be written as $\operatorname{rank} A \leq 1$, where

$$A = \begin{pmatrix} x_0 & \dots & x_{a-1} & \dots & x_{a+b} & \dots \\ x_1 & \dots & x_a & \dots & x_{a+b+1} & \dots \end{pmatrix}.$$

If $X \subset F$ is residual to a number of generators of the ruling of F, then it is clear that the defining equations of X are $\operatorname{rank} A \leq 1$ together with a set of equations, essentially just one equation with rolling factors corresponding to the residual linear system.

For example, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ is defined by $\operatorname{rank} A \leq 1$, where $A = (a_{ij})$ is a generic 2×4 matrix. The free resolution of the ideal defining F is of the form

$$0 \to 3\mathcal{O}(-4) \to 8\mathcal{O}(-3) \to 6\mathcal{O}(-2) \to 0,$$

where the maps are given by determinantal syzygies that can easily be written out explicitly. One sees from this that the general anticanonical divisor of F is defined by 3 quartics $f_1,\ f_2,\ f_3$ satisfying $(f_1, f_2, f_3)N = 0$ on F, where

$$N = \begin{pmatrix} a_{11} & -a_{12} & a_{13} & -a_{14} & & & & \\ -a_{21} & a_{22} & -a_{23} & a_{24} & a_{11} & -a_{12} & +a_{13} & -a_{14} \\ & & & & -a_{21} & a_{22} & -a_{23} & a_{24} \end{pmatrix}.$$

Assuming the a_{ij} are independent indeterminates, these can be solved to see that f_1 is a sum of terms involving at least 2 factors from the top row of A, and $f_2,\ f_3$ are obtained by rolling factors. By the way, taking a general anticanonical divisor of a codimension 3 Cohen–Macaulay variety is an obvious surefire way of getting a codimension 4 Gorenstein variety.

Another variant on the $2 \times n$ determinantal is the notion of *quasi-determinantal* due to Riemenschneider: consider the quasimatrix (or crazy matrix?)

$$A = \begin{pmatrix} x_1 & & x_2 & & \dots & & x_i & & \dots & & x_j & & \dots \\ & u_1 & & u_2 & & u_{i-1} & & u_i & & u_{j-1} & & u_j & \\ y_1 & & y_2 & & \dots & & y_i & & \dots & & y_j & & \dots \end{pmatrix};$$

by definition, for $i < j$ the ijth minor of A is $x_i y_j - y_i u_i \dots u_{j-1} x_j$. In other words, to evaluate a minor one multiplies in the usual way when going southeast, but one must pay to go northeast across a block of squares by multiplying by the product of the indicated charges. Riemenschneider showed that this is a flexible format [Rie], and that every surface quotient singularity can be written in this way. This has lots of applications in the deformation theory of surface quotient singularities (see [S] and the references given there).

5.3 Pfaffians with extra symmetry

I know of many different formats all of which give rise to Gorenstein rings in codimension 4 defined by 9 equations yoked by 16 syzygies. For example the determinantal equations $\operatorname{rank} M \le 1$ where M is a 3×3 matrix; if the entries of M are general linear forms in $\mathbb{P}^8$ then these are the equations defining the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. Consider however the following trick: let $M = A + B$ with A symmetric and B skew, and write

$$P = \begin{pmatrix} B & A \\ -A & -B \end{pmatrix};$$

an easy calculation shows that the ideal of 2×2 minors of M coincides with the ideal of 4×4 diagonal Pfaffians of P. However, the 6×6 extra-symmetric Pfaffian format can be generalised, for example by taking $P' = \left(\begin{smallmatrix} B & A \\ -A & -uB \end{smallmatrix}\right)$ for some factor u; it is easy to see that P' also defines a codimension 4 Gorenstein variety defined by 9 equations yoked by 16 syzygies. If u is not a square, then the Pfaffian format cannot be converted back to a 3×3 determinant. I want to consider P as a deformation of P' obtained by letting u tend to 1, but this does not seem to make sense formally.

Under what conditions does it happen that a 6×6 skew matrix P gives rise to a codimension 4 Gorenstein variety? I do not know. For it to happen, 6 of the Pfaffians must be linear combinations of the others; the following example seems to show that this can happen without any vestige of extra symmetry.

In studying Theorem 5, Dicks makes the ingenious observation that there is a way of cooking up the set of 9 equations given there from the 4×4 diagonal Pfaffians of the following beautiful 6×6 skew matrix:

$$\begin{pmatrix} 0 & 1 & x_1 & y_1 & x_2^2 & z_1 \\ & 0 & x_2 & x_1^2 & y_2 & z_2 \\ & & 0 & -rz_1+s^2x_1y_2 & -sz_2-r^2x_2y_1 & -ry_1^2-sy_2^2 \\ & & & & \text{*** see below ***} & \\ & -\text{sym} & & & & \end{pmatrix}$$

where the outsize bottom 3×3 block is

$$\begin{pmatrix} 0 & -sx_1z_1+rx_2z_2-rsh & r(x_1h+x_2y_2^2)-sx_1x_2^2y_2 \\ & 0 & -rx_1^2x_2y_1+s(x_1y_1^2+x_2h) \\ -\text{sym} & & 0 \end{pmatrix}.$$

This procedure is quite mysterious: 6 of the 15 Pfaffians give the first 6 relations in the obvious way; the remaining 9 are all in the ideal generated by the 9 relations, but with r and s as coefficients; for example,

$$\begin{aligned} (13:46) &= rR_7 + sy_2R_3, \\ (13:56) &= sR_8 - rx_2y_1R_1, \\ (23:56) &= sR_9 + sx_1y_1R_1 - ry_1R_3. \end{aligned}$$

The relations R_7, R_8, R_9 thus only appear after cancelling a factor, so that the Pfaffians as they stand do not define the deformation family (they go wrong when r or $s = 0$). This construction seems to force the syzygies, but I do not know how to prove this.

The need to cancel factors before getting the right relations is strongly reminiscent of what happens if one tries to force the quasideterminantal equations naively into a simple determinantal form; maybe there is a notion of crazy Pfaffian analogous to crazy determinantals trying to materialise.

5.4 Where to go from here?

I believe that there are structure theorems on Cohen–Macaulay or Gorenstein rings in small codimension under suitable extra conditions, or at least common generalisations of the existing mess of examples. My hope is to get more experience with these types of rings and their deformation theory. There are really hundreds of examples: weighted cones over K3s, Gorenstein cyclic quotient singularities in dimension 3 or 4, the general anticanonical divisor of a quasideterminantal 3-fold, etc; and it will probably be easier to see through the fog when some of these have been given the infinitesimal treatment (preferably by machine).

5.5 Two final problems

Graded rings corresponding to halfcanonical linear systems on hyperelliptic curves have such a beautiful and conclusive description (see [R2], Section 4) that one yearns for generalisations to surfaces. However, except possibly for a few initial cases, this is likely to be hard.

5.5.1 The canonical ring of a genus 2 pencil

Let S be a regular surface with a genus 2 pencil $\varphi\colon S \to C = \mathbb{P}^1$. The local structure of the relative canonical algebra $\mathcal{R}(\varphi) = \bigoplus \varphi_* \omega_{S/C}^{\otimes k}$ is well understood: it is of the form $\mathcal{O}_C[x_1, x_2, z]/(f_6)$ (that is, a double cover of $\mathbb{P}^1$) near a 2-connected fibres and $\mathcal{O}_C[x_1, x_2, y, z]/(q_2(x_1, x_2), f_6)$ near a 2-disconnected fibres (that is, a double cover of the line pair or double line $(q_2 = 0) \subset \mathbb{P}(1^2, 2)$); see Section 4 for an example of a 2-disconnected fibre. Globally, $\varphi_*\omega_{S/C} = \mathcal{O}_{\mathbb{P}^1}(a_1, a_2)$ and $\varphi_*\omega_{S/C}^{\otimes 2} = \mathcal{O}_{\mathbb{P}^1}(b_1, b_2, b_3)$ are also easy to handle, but the multiplication map $S^2(\varphi_*\omega) \to \varphi_*\omega^{\otimes 2}$ is subtle and contains all the information on the 2-canonical image of S, that is, the conic bundle $X/i \to C$, where X is the canonical model and i its biregular hyperelliptic involution.

Thus the canonical ring of S should have a nice description, in terms of two data, the geometry of a conic bundle and rolling factors; the latter appear if you twist back the bundles $\varphi_*\omega$, $\varphi_*\omega^{\otimes 2}$ and the antiinvariant part of $\varphi_*\omega^{\otimes 3}$ to get global bases.

5.5.2 Hyperelliptic surfaces with $p_g = 3$

Suppose S is a surface of general type for which the general canonical curve $(C \in |K_S|, \mathcal{O}_C(1) = K_{S|C})$ is nonsingular and hyperelliptic, polarised by $g^1_2 + P_1 + \cdots + P_k$ with the P_i Weierstrass points; of course $K_S^2 = k + 2$ and $g(C) = K_S^2 + 1 = k + 3$. The cases $k = 0, 1$ are classical by Enriques and Horikawa, and $k = 2$ has been the subject of this lecture. For higher k one does not necessarily aspire to such precise results, and for $k \geq 12$ or so things presumably become impossibly difficult.

The 1-canonical map $\varphi_K \colon S \dashrightarrow \mathbb{P}^2$ blows up the k points P_i and maps them to an arrangement of k distinct lines $\ell_i \subset \mathbb{P}^2$; birationally, φ is a double cover with branch locus $\bigcup \ell_i$ together with a plane curve B of degree $2g + 2 - k = k + 8$ with singular points of given multiplicity on the lines ℓ_i and at the multiples points of the arrangement. Already for $k = 3$, and with everything generic, there are two different combinatorial possibilities for the branch locus: 3 nonconcurrent lines, and B has a triple point on each ℓ_i and a 4-ple point at each vertex $\ell_i \cap \ell_j$; or 3 concurrent lines, and B has two triple point on each ℓ_i and a 5-ple point at $\ell_1 \cap \ell_2 \cap \ell_3$

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