A simply connected surface of general type with $p_g = 0$, $K^2 = 1$, due to Rebecca Barlow

(notes by M. Reid)

March 1981

0 Introduction

Suppose that X is the canonical model of a surface of general type with $p_g = 0$, $K^2 = 1$; assume that X has an even set $\{P_1, \dots, P_4\}$ of 4 nodes, and hence a double cover $Y \to X$ ramified in just these 4 nodes. Then Y has $p_g = 0$, $K^2 = 2$, and hence $|\pi_1 Y| \le 9$, but a priori one does not know what π_1 will be.

Let $F \to Y$ be the universal cover of Y; then $F \to X$ is Galois with group G, and $\pi_1 Y \triangleleft G$ has $G/\pi_1 Y \cong \mathbb{Z}/2$. Since X has only nodes, the elements of G that have fixed points on F are necessarily involutions; call these the *elliptic* elements. They generate a normal subgroup $E \triangleleft G$, and $\pi_1 X = G/E$.

If $\pi_1 X = \{0\}$, Barlow [B] uses a straightforward group theoretic argumnet to show that $|\pi_1 Y|$ cannot be even. This leaves the possibilities $\{0\}$, $\mathbb{Z}/3$, $\mathbb{Z}/5$, $\mathbb{Z}/7$ or $|\pi_1 Y| = 9$. The last two cases seem rather implausible, but one guesses that the first 3 cases could occur.

Barlow then shows that $\pi_1 Y = \mathbb{Z}/5$ can occur (see [B1]). For this she needs to construct a (nonsingular, simply connected) surface F with $p_g = 4$, $K^2 = 10$, and an action on F of the dihedral group D_{10}, in such a way that the normal subgroup $\mathbb{Z}/5 \subset D_{10}$ acts freely, and each of the 5 conjugate involutions of D_{10} has just 4 isolated fixed points. One checks at once that the quotient $X = F/D_{10}$ has the required properties.

Barlow's construction leads to a family of examples (apparently) depending on 4 moduli. I give here a particular example, based on a surface F having an action of $\mathbb{Z}/2 \times \mathfrak{S}_5$ that has already been considered in detail in the literature; see especially [C]. I am endebted to Derek Holt for the advice to exorcise the Young tableaux from [C], and for the superior description of the representation W in Section 3, which leads to a considerable tidying up of the construction in [C].

1 The surface F and its canonical ring

Let F be a surface with $p_g = 4$, $q = 0$, $K^2 = 10$, for which $\varphi_K \colon F \to Q \subset \mathbb{P}^3$ is a double cover of a quintic, ramified in just 20 nodes of Q. The double cover $F \to Q$ has a covering involution $i \colon F \to F$ that acts on the canonical ring $R = \bigoplus_{n\geq 0} H^0(\mathcal{O}_F(nK_F))$, decomposing it as the sum of $R^+ = R(Q)$ and R^-. According to [C, Theorem 3.3], on choosing a basis

$$x_1, \dots, x_4 \in R_1 \quad \text{and} \quad y_1, \dots, y_5 \in R_2^-,$$

we get $R = k[x_1, \dots, x_4, y_1, \dots, y_5]/I$, where the ideal I of relations can be described in terms of a symmetric 5×5 matrix A with entries linear forms in the x_i. Thus I is generated by

$$\begin{array}{lll} \text{in degree 3,} & \sum A_{ij} y_j & \text{(5 values of } i\text{)} \\ \text{in degree 4,} & y_i y_j - B_{ij} & \text{(15 values of } (ij)\text{)}, \end{array}$$

where B_{ij} is the ijth 4×4 minor of A. Note that I automatically contains $\det A$, $B_{ik} y_j - B_{jk} y_i$, etc. The quintic $Q \subset \mathbb{P}^3$ is defined by $\det A$, and $F = \operatorname{Proj} R$ is a double cover $F \to Q$. Thus F is nonsingular, and $F \to Q$ ramified in just 20 nodes of Q for general values of the entries of A.

2 A group action on F

If a finite group G acts on F then it will act on any vector space canonically associated with F; in particular, G will have a representation r_1 on R_1, r_2 on R_2^-, and r_3 on

$$\ker\{R_1 \otimes R_2^- \to R_3^-\},$$

which is the 5-dimensional vector space based by the 5 relations $\sum A_{ij} y_j$. The final representation r_3 is given as follows: if $g \in G$ acts on the relation $\sum A_{ij} y_j$ we get a new relation

$$\sum r_1(g)(A_{ij}) \cdot r_2(g) y_j;$$

(recall that A_{ij} is a linear form in the x_i). This is a new relation between the elements $x_i y_j \in R_3^-$, so is a linear combination of the $\sum A_{ij} y_j$:

$$\sum r_1(g)(A_{ij}) \cdot r_2(g) y_j = r_3(g) \left(\sum A_{ij} y_j \right).$$

In matrix terms

$$r_3(g)^{-1} \cdot r_1(g)(A) \cdot r_2(g) = A. \qquad (*)$$

In practice one can usually predict in advance the representation r_1, r_2, r_3 by character theory; for the matrix A to be symmetric we have to coordinate our choice of bases in R_2^- and ker, and for this to work, it is highly desirable that $r_3 = {}^t r_2^{-1}$.

The next two sections show how to pick r_1, r_2, r_3 and A to satisfy $(*)$.

3 Irreducible 4 and 5-dimensional representations of $\mathfrak{S}_5$

The following result is well known:

Proposition *Suppose that a group G acts doubly transitively on a finite set T; then the permutation representation $\sum_{t\in T} k \cdot T$ decomposes as $I \oplus U$, where I is the trivial 1-dimensional representation spanned by $\sum t$, and U is irreducible.*

Applying this to $G = \mathfrak{S}_5$, $T = \{1, 2, 3, 4, 5\}$ we get a 4-dimensional representation V with spanning set $x_1, \ldots, x_5$ subject to the single relation $\sum x_i = 0$.

$\mathfrak{S}_5$ also acts by conjugation on the set consisting of its 6 subgroups of order 5:

$$H_\infty = \langle (12345) \rangle, \quad H_0 = \langle (12543) \rangle, \quad H_1 = \langle (12534) \rangle,$$

$$H_2 = \langle (12435) \rangle, \quad H_3 = \langle (12354) \rangle, \quad H_4 = \langle (12453) \rangle;$$

the subscripts are to be thought of as points $\alpha \in \mathbb{P}^1(\mathbb{F}_5)$, and the action gives the sporadic isomorphism $\mathfrak{S}_5 \cong \mathrm{PGL}(2, \mathbb{F}_5)$. One checks that on generators

$$\begin{aligned} (12345): &\quad \alpha \mapsto \alpha + 1, \\ (12): &\quad \alpha \mapsto 2/\alpha. \end{aligned}$$

We thus get an irreducible 5-dimensional representation W of $\mathfrak{S}_5$, with spanning set $y_\infty, y_0, \ldots, y_4$ subject to the single relation $\sum y_\alpha = 0$.

4 An invariant element of $V \otimes S^2 W$

Character theory implies that the trivial representation of $\mathfrak{S}_5$ appears just once in $V \otimes S^2 W$. One can check directly that the following element is invariant under $\mathfrak{S}_5$:

$$\begin{aligned}\ell = x_1(y_0y_4 + y_1y_3 + y_2y_\infty) + x_2(y_0y_1 + y_2y_4 + y_3y_\infty)\\ + x_3(y_0y_3 + y_1y_2 + y_4y_\infty) + x_4(y_0y_\infty + y_1y_4 + y_2y_3)\\ + x_5(y_0y_2 + y_1y_\infty + y_3y_4).\end{aligned}$$

Alternatively, the invariance can be proved without any computation, as follows: every subgroup $H_\alpha \subset H_5$ of order 5 is contained in a unique conjugate D_α of $D_{10} = \langle(12345), (25)(34)\rangle$. Any two distinct D_α and D_β contain a unique conjugate of $(25)(34)$; finally, the monomial $x_iy_\alpha y_\beta$ occurs in ℓ if and only if $(jk)(lm) \in D_\alpha \cap D_\beta$, where $\{i, j, k, l, m\}$ is a permutation of $\{1, 2, 3, 4, 5\}$.

This element can be written

$$\ell = (y_0, \ldots, y_4, y_\infty) \begin{pmatrix} 0 & x_2 & x_5 & x_3 & x_1 & x_4 \\ & 0 & x_3 & x_1 & x_4 & x_5 \\ & & 0 & x_4 & x_2 & x_1 \\ & \text{(sym)} & & 0 & x_5 & x_2 \\ & & & & 0 & x_3 \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_4 \\ y_\infty \end{pmatrix}$$

Setting $y_\infty = -\sum_{\alpha=0}^{4} y_\alpha$, this becomes

$$\begin{aligned}\ell &= (y_0, \ldots, y_4) \begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix} \begin{pmatrix} \text{(same} \\ \text{matrix)} \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_4 \end{pmatrix} \\ &= (y_0, \ldots, y_4) A \begin{pmatrix} y_0 \\ \vdots \\ y_4 \end{pmatrix},\end{aligned}$$

where

$$\begin{pmatrix} -2x_4 & x_2 - x_5 - x_4 & x_5 - x_1 - x_4 & x_3 - x_2 - x_4 & x_1 - x_3 - x_4 \\ & -2x_5 & x_3 - x_1 - x_5 & x_1 - x_2 - x_5 & x_4 - x_3 - x_5 \\ & & -2x_1 & x_4 - x_2 - x_1 & x_2 - x_3 - x_1 \\ & \text{(sym)} & & -2x_2 & x_5 - x_3 - x_2 \\ & & & & -2x_3 \end{pmatrix}$$

By construction, this matrix has the covariance

$$ {}^t r_2(g) \cdot r_1(g)(A) \cdot r_2(g) = A \quad \text{for } g \in \mathfrak{S}_5, \qquad (**)$$

where $r_1(g)$ acts on V by permuting the x_i and $r_2(g)$ acts on W by permuting the y_α. In terms of the basis $\{y_0, \dots, y_4\}$ of W, r_2 is the representation

$$r_2(12345) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad r_2(12) = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

5 The quintic Q

Because of the covariance $(**)$, $\det A$ is a quintic symmetric in the x_i. It follows from the theory of elementary symmetric functions that $\det A = \lambda S_5 + \mu S_2 S_3$ modulo $S_1 = \sum x_i$, for some $\lambda, \mu \in \mathbb{Q}$, where $S_k = \sum x_i^k$ are the power sums. By evaluating both sides at $(2, -1, -1, 0, 0)$ and $(3, -2, -1, 0, 0)$ (say), one sees that

$$\det A = \frac{36}{5}(4S_5 - 5S_2S_3).$$

One can verify directly (or see [GZ, p. 104]) that $\det A$ defines a quintic $Q \subset \mathbb{P}^3$ having 20 nodes at $(2, 2, 2, -3 + \sqrt{-7}, -3 - \sqrt{-7})$ and its $\mathfrak{S}_5$ translates, and no other singularities.

One also checks that the involution $(25)(34)$ acts on Q with a fixed locus consisting of the line $\ell : x_1 = x_2 + x_5 = x_3 + x_4 = 0$, together with the 5 isolated fixed points:

$$x_2 = x_5, \quad x_3 = x_4, \quad (x_2 + x_3)(3x_2 + x_3)(x_2 + 3x_3)(x_2^2 + x_2x_3 + x_3^2) = 0.$$

6 An action of D_{10} on F

The group $\mathbb{Z}/2 \times \mathfrak{S}_5$ now acts on R, and hence on F: the involution $i \times 1$ acts by

$$x_i \mapsto x_i \quad \text{and} \quad y_i \mapsto -y_i,$$

whereas $\mathfrak{S}_5$ acts by permuting the x_i and y_α as in Section 3. The fixed points of any $g \in \mathbb{Z}/2 \times \mathfrak{S}_5$ acting on F lie over the fixed points of the second factor g_2 acting on Q.

Consider the subgroup $D \subset \mathbb{Z}/2 \times \mathfrak{S}_5$ generated by $1 \times (12345)$ and $i \times (25)(34)$. Obviously, D is isomorphic to D_{10}. It is easy to check that (12345) acts freely on Q (see [C, Section 4]), so that the normal $\mathbb{Z}/5 \subset D$ acts freely on F.

Finally, we have to check that $i \times (25)(34)$ acts on F with just 4 fixed points. The element $(25)(34) \in \mathfrak{S}_5$ acts on $\mathbb{P}^1(\mathbb{F}_5)$ as $\alpha \mapsto 4-\alpha$. One checks that over the line $L : x_1 + x_4 = x_2 + x_3 = x_5 = 0$, the cover $F \to Q$ splits into two components L_1 and L_2, with

$$L_1 : \quad \begin{aligned} y_0 = y_4 &= 3x_2^2 + 2x_2x_3 + 3x_3^2, \\ y_1 = y_3 &= 3x_2^2 - 2x_2x_3 + 3x_3^2, \\ y_5 &= -3x_2^2 - 4x_2x_3 + 3x_3^2 \end{aligned}$$

and L_2 obtained by reversing the sign of each y_α. It follows that $(25)(34)$ fixes each of L_1 and L_2 pointwise, and that $i \times (25)(34)$ interchanges them.

This means that our involution $i \times (25)(34)$ has at most 10 fixed points on F, lying over the line $m : x_2 = x_5, x_3 = x_4$ of $\mathbb{P}^3$. The reader can check as an exercise that this is already enough to guarantee that it then has exactly 4 fixed points. Alternatively, argue as follows: over the point $x_1 = -2(u+v)$, $x_2 = x_5 = u$, $x_3 = x_4 = v$, the matrix A becomes

$$A_{|m} = \begin{pmatrix} -2v & -v & v+3u & -u & -2u-4v \\ & -2u & u+3v & -4u-2v & -u \\ & & 4(u+v) & u+3v & v+3u \\ & \text{(sym)} & & -2u & -v \\ & & & & -2v \end{pmatrix}$$

This matrix has an unexpected symmetry about the antidiagonal. Subtracting the 5th row from the 1st and the 4th row from the 2nd gives

$$\begin{aligned} 2(u+v)(y_0 - y_4) &+ (u-v)(y_1 - y_3) &= 0, \\ (u-v)(y_0 - y_4) &+ 2(u+v)(y_1 - y_3) &= 0. \end{aligned}$$

It follows that, on the line m and outside the zeros of the determinant $4(u+v)^2 - (u-v)^2 = (3u+v)(u+3v)$, the corresponding point of F has $y_0 = y_4$ and $y_1 = y_3$. Therefore the inverse images of the 3 points $(u+v)(u^2+uv+v^2)$ are fixed by (25)(34) and not fixed by $i \times (25)(34)$.

It's easy to see that the 4 inverse images of the 2 points $(3u+v)(u+3v)$ are indeed fixed by $i \times (25)(34)$, and this completes the construction.

References

[B] Rebecca Barlow, Some new surfaces with $p_g = 0$, Univ. of Warwick Ph.D. thesis, Sep. 1982, 90 + vii pp.

[B1] Rebecca Barlow, A simply connected surface of general type with $p_g = 0$, Invent. Math. **79** (1985) 293–301

[B2] Rebecca Barlow, Some new surfaces with $p_g = 0$, Duke Math. J. **51** (1984) 889–904

[B3] Rebecca Barlow, Rational equivalence of zero cycles for some more surfaces with $p_g = 0$, Invent. Math. **79** (1985) 303–308

[B4] Rebecca Barlow, Complete intersections and rational equivalence, Manuscripta Math. **90** (1996) 155–174

[B5] Rebecca Barlow, Zero-cycles on Mumford's surface, Math. Proc. Cambridge Philos. Soc. **126** (1999) 505–510

[C] Fabrizio Catanese, Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications, Invent. Math. **63** (1981) 433–465

[GZ] G. van der Geer and D. Zagier, The Hilbert modular group for the field $\mathbb{Q}(\sqrt{13})$, Invent. Math. **42** (1977) 93–133

Miles Reid,
Math Inst., Univ. of Warwick,
Coventry CV4 7AL, England
e-mail: miles@maths.warwick.ac.uk
web: www.maths.warwick.ac.uk/~miles